

Exact Kähler Potential for Calabi-Yau Fourfolds

Yoshinori Honma^{*} and Masahide Manabe[†]

Harish-Chandra Research Institute

Chhatnag Road, Jhusi, Allahabad 211019, India

Abstract

We study quantum Kähler moduli space of Calabi-Yau fourfolds. Our analysis is based on the recent work by Jockers et al. which gives a novel method to compute the Kähler potential on the quantum Kähler moduli space of Calabi-Yau manifold. In contrast to Calabi-Yau threefold, the quantum nature of higher dimensional Calabi-Yau manifold is yet to be fully elucidated. In this paper we focus on the Calabi-Yau fourfold. In particular, we conjecture the explicit form of the quantum-corrected Kähler potential. We also compute the genus zero Gromov-Witten invariants and test our conjecture by comparing the results with predictions from mirror symmetry. Local toric Calabi-Yau varieties are also discussed.

^{*}yhonma@hri.res.in

[†]masahidemanabe@gmail.com

1 Introduction

Topological string theory gives us a framework to study “topological quantum quantities” as Gromov-Witten invariants corresponding to the worldsheet instanton numbers. Strongly motivated from the string compactification, Calabi-Yau threefolds has been well-studied. The quantum-corrected Kähler potential on the Kähler moduli space of Calabi-Yau threefold can be determined by a single holomorphic function called prepotential which encodes the information about the genus zero Gromov-Witten invariants. This simplification is due to the special Kähler structure of the moduli space of Calabi-Yau threefold [1]. Mirror symmetry is an efficient tool to compute the prepotential as first demonstrated in [2] to the quintic Calabi-Yau threefold. Although moduli spaces of Calabi-Yau manifolds whose complex dimension is greater than three do not have such a special Kähler structure, the higher dimensional mirror symmetry is still useful to compute Gromov-Witten invariants [3, 4, 5] (see also [6]). As pioneered in [7], various Calabi-Yau manifolds can be realized at the IR fixed points of the two dimensional $\mathcal{N} = (2, 2)$ gauged linear sigma model (GLSM). Mirror symmetry for Calabi-Yau manifolds described by abelian GLSMs is well understood [8, 9] and also proved physically in [10]. However, for general cases with non-abelian gauge groups, comprehensive study of mirror symmetry is yet to come.

Recently, it was proposed in [11] that Gromov-Witten invariants of Calabi-Yau manifolds can be computed without using mirror symmetry. More precisely, they proposed the relation

$$e^{-K} = Z_{\text{GLSM}}$$

between the exact Kähler potential K on the quantum Kähler moduli space of a Calabi-Yau manifold and the exact partition function Z_{GLSM} of an $\mathcal{N} = (2, 2)$ GLSM on S^2 which was computed in [12, 13]. This means that we can extract the genus zero Gromov-Witten invariants only from the GLSM calculation. They also checked the consistency of their proposal for some known examples and two physical proofs are given in [14]. Their method is applicable to non-abelian GLSMs and the authors of [11] also made predictions for Gromov-Witten invariants of a determinantal Calabi-Yau threefold. Note that, as mentioned in [11, 14], their method has a close relationship to the toric mirror symmetry.

The aim of this paper is to study the quantum Kähler moduli space of Calabi-Yau fourfold using the novel method of [11]. We conjecture a formula of the exact Kähler potential in (3.3) and provide a prescription to compute the Gromov-Witten invariants of Calabi-Yau fourfolds. We test our conjecture by computing the Gromov-Witten invariants of several Calabi-Yau fourfolds and comparing them with known results in the literature.

We also study local toric Calabi-Yau varieties, and propose a correspondence in (5.2) which precisely express a local toric analogue of the statement of [11].

This paper is organized as follows. In Section 2, we first review the mirror symmetry in general dimension. It gives an introduction to the Gromov-Witten theory from the aspects of mirror symmetry. Then we summarize the proposal of [11] with the results of [12, 13]. In Section 3, we represent our main result about the quantum Kähler moduli space for Calabi-Yau fourfolds, and refer to the relationship to the mirror manifold. In Section 4, we test our proposal by demonstrating the exact GLSM calculation for several examples of compact Calabi-Yau fourfolds. In Section 5, we consider local toric Calabi-Yau varieties and some remarks about the non-compact limit are given. Section 6 is devoted to the conclusion and discussion. We summarize the results of general complete intersections in the Grassmannian in Appendix A.

2 Some known results

In this section, we collect some known results on mirror symmetry of higher dimensional Calabi-Yau manifolds [3, 4, 5]. Then we summarize the proposal in [11] which gives a new method to compute the exact Kähler potential on the quantum Kähler moduli space of Calabi-Yau manifolds via two dimensional $\mathcal{N} = (2, 2)$ GLSM partition function.

2.1 Moduli spaces and mirror symmetry

As is well known, one can define two types of two dimensional topological field theories, the A-model and the B-model, by gauging either the vector $U(1)_V$ or the axial vector $U(1)_A$ R-symmetries in the $\mathcal{N} = (2, 2)$ non-linear sigma model [15, 16] (see also [10]). The A-model on a Calabi-Yau d -fold X without boundary depends only on the Kähler moduli of X and is independent of the complex structure moduli. On the other hand, the B-model on a Calabi-Yau d -fold X^* without boundary only captures the information about the complex structure moduli of X^* . By Batyrev's mirror construction for Calabi-Yau complete intersections in toric varieties [17, 18], one can construct a number of mirror pairs (X, X^*) satisfying

$$\dim H^i(\wedge^j T^* X) = \dim H^i(\wedge^j T X^*), \quad \dim H^i(\wedge^j T X) = \dim H^i(\wedge^j T^* X^*), \quad 0 \leq i, j \leq d, \quad (2.1)$$

where $T^* X$ and $T X$ are holomorphic cotangent and tangent bundles on X , respectively. The Kähler moduli space $\mathcal{M}_{\text{Kähler}}(X)$ of X is locally isomorphic to $H^1(\wedge^1 T^* X)$, while the

complex structure moduli space $\mathcal{M}_{\text{comp}}(X^*)$ of X^* corresponds to $H^1(\wedge^1 TX^*)$. Therefore the mirror symmetry implies the existence of the isomorphism between $\mathcal{M}_{\text{Kähler}}(X)$ and $\mathcal{M}_{\text{comp}}(X^*)$. Moreover the mirror symmetry yields the equivalence of the A-model correlators of the observables $\mathcal{O}_\ell^{(i)}$ defined by the elements $J_\ell^{(i)}$ in $H^i(\wedge^i T^*X)$ and the B-model correlators of the observables $\tilde{\mathcal{O}}_\ell^{(i)}$ corresponding to $\tilde{J}_\ell^{(i)}$ in $H^i(\wedge^i TX^*)$. Here the index ℓ just labels the elements. As a consequence of the Kähler moduli dependence, the A-model correlators receive α' corrections corresponding to the worldsheet instantons, whereas the B-model correlators do not receive these kind of corrections. Therefore the mirror symmetry provides a powerful method to compute the quantum corrections in the A-model by means of the classical calculation of the B-model.

As shown in [3], the three-point A-model correlator on the worldsheet \mathbb{P}^1 takes the form

$$C_{\ell mn}^{(i,j)} = \langle \mathcal{O}_\ell^{(i)} \mathcal{O}_m^{(j)} \mathcal{O}_n^{(d-i-j)} \rangle_{\mathbb{P}^1} = \kappa_{\ell mn}^{(i,j)} + \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} N_{\beta, \ell mn}^{(i,j)} \frac{q^\beta}{1 - q^\beta}, \quad (2.2)$$

where $q^\beta \equiv e^{2\pi i \sum_{\ell=1}^{h^{1,1}(X)} \beta_\ell t^\ell}$ and t^ℓ are the complexified Kähler parameters.¹ The leading contributions $\kappa_{\ell mn}^{(i,j)} = \int_X J_\ell^{(i)} \wedge J_m^{(j)} \wedge J_n^{(d-i-j)}$ are the classical intersection numbers and the coefficients $N_{\beta, \ell mn}^{(i,j)}$ correspond to the genus zero Gromov-Witten invariants related to the number of rational curves (the worldsheet instantons) of class β which intersect with the cycles dual to the inserted observables. In particular, it is thought that

$$n_{\beta, n} = \frac{N_{\beta, \ell mn}^{(1,1)}}{\beta_\ell \beta_m} \quad (2.3)$$

are integer invariants [6].² The generating function of the Gromov-Witten invariants $n_{\beta, n}$ can be defined as

$$F_n(t) = \frac{1}{2} \sum_{\ell, m=1}^{h^{1,1}(X)} \kappa_{\ell mn}^{(1,1)} t^\ell t^m + \hat{F}_n(t), \quad \hat{F}_n(t) = \frac{1}{(2\pi i)^2} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_{\beta, n} \text{Li}_2(q^\beta), \quad (2.4)$$

where $\text{Li}_k(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^k}$ is the polylogarithm. Then the three-point correlation function (2.2) with $i = j = 1$ can be expressed as $C_{\ell mn}^{(1,1)} = \frac{\partial}{\partial t^\ell} \frac{\partial}{\partial t^m} F_n(t)$.³ Here we have used an abridged notation $t = \{t^\ell\}_{\ell=1}^{h^{1,1}}$ for the complexified Kähler moduli.

¹This multiple cover formula of the three-point correlator is the higher dimensional analogue of the Aspinwall-Morrison formula for Calabi-Yau threefold [19]. In the case of the Calabi-Yau threefold, $n_\beta \equiv \frac{N_{\beta, \ell mn}^{(1,1)}}{\beta_\ell \beta_m \beta_n}$ can be interpreted as the genus zero part of the Gopakumar-Vafa invariants which count the BPS states of M2 branes wrapping the holomorphic two-cycles [20, 21].

²Note that these invariants are independent of the observables $\mathcal{O}_\ell^{(1)}$ and $\mathcal{O}_m^{(1)}$.

³As discussed in [5] from the perspective of mirror B-model, $C_{\ell mn}^{(1,1)}$ provides all the three-point correlators $C_{\ell mn}^{(i,j)}$. Observables of these correlators are restricted to take values at the primary subspace of the vertical cohomology $\oplus_i H^{i,i}(X)$ which is generated by wedge products of the elements of $H^{1,1}(X)$ [3].

By using the mirror symmetry, the holomorphic functions $F_n(t)$ at the large radius point $q = 0$ can be determined from solutions to Picard-Fuchs equations which govern the periods of the holomorphic d -form $\Omega(z)$ on X^* . Here z indicates the set of the complex structure moduli $z = \{z_\ell\}_{\ell=1}^{h^{d-1,1}}$. These equations can be derived from the theory of variation of the Hodge structures [22, 3] and the solutions to the equations reflect the structure of primary subspace of the horizontal cohomology $\oplus_i H^{d-i,i}(X^*)$ generated by the cup products of the elements in $H^{d-1,1}(X^*)$ [3]. At the large complex structure point $z = 0$, there are $h_{\text{prim}}^{d-i,i}(X^*)$ ($0 < i < d$) linearly independent solutions $\Pi_\ell^{(i)}(z)$ whose leading contributions are the i -th power of the logarithm of z . Then the flat coordinates of the complex structure moduli space $\mathcal{M}_{\text{comp}}(X^*)$ identified with the complexified Kähler moduli of X are determined by the mirror map

$$t^\ell = \frac{\Pi_\ell^{(1)}(z)}{2\pi i \Pi^{(0)}(z)} \sim \frac{1}{2\pi i} \log z_\ell, \quad (2.5)$$

where $\Pi^{(0)}(z) = 1 + \mathcal{O}(z)$ is a unique single-valued solution. Through the use of the mirror map, the holomorphic function (2.4) corresponding to the observable $\mathcal{O}_n^{(d-2)}$ restricted to the primary subspace of $H^{d-2,d-2}(X)$ can be derived from a period as

$$F_n(t) = \frac{\Pi_n^{(2)}(z)}{(2\pi i)^2 \Pi^{(0)}(z)} \sim \frac{1}{2(2\pi i)^2} \sum_{\ell,m=1}^{h^{d-1,1}(X^*)} \kappa_{\ell mn}^{(1,1)} \log z_\ell \log z_m, \quad (2.6)$$

where $\kappa_{\ell mn}^{(1,1)}$ are identified with the classical intersection numbers of X .

For Calabi-Yau threefold, both the Kähler and the complex structure moduli spaces are the local special Kähler manifold [1, 23, 24, 25]. The Kähler potential of such a manifold can be determined by the prepotential $\mathcal{F}(T)$ satisfying the scaling property $\mathcal{F}(\lambda T) = \lambda^2 \mathcal{F}(T)$, where T is a set of the special coordinates of the moduli space defined by $T = \{T^0, T^\ell\} \equiv \{\Pi^{(0)}, \Pi^{(0)} t^\ell\}$. At the large complex structure point, the log-squared solutions $\Pi_\ell^{(2)}(z)$ correspond to $\mathcal{F}_\ell(T) \equiv \frac{\partial}{\partial T^\ell} \mathcal{F}(T)$, and a log-cubed solution $\Pi^{(3)}(z) \sim -\frac{1}{3!(2\pi i)^3} \sum_{\ell,m} \kappa_{\ell mn}^{(1,1)} \log z_\ell \log z_m \log z_n$ becomes equivalent to $\mathcal{F}_0(T) \equiv \frac{\partial}{\partial T^0} \mathcal{F}(T)$. The solutions $\{\mathcal{F}_0, \mathcal{F}_\ell\}$ are also known as the conjugate special coordinates of the moduli space. Using the scaling property of the prepotential, one can define $\mathcal{F}(T) = (T^0)^2 F(t)$ and the conjugate coordinates can be expressed in terms of t^ℓ as

$$\mathcal{F}_\ell(T) = T^0 \frac{\partial}{\partial t^\ell} F(t), \quad \mathcal{F}_0(T) = T^0 \left[2F(t) - \sum_{\ell=1}^{h^{1,1}(X)} t^\ell \frac{\partial}{\partial t^\ell} F(t) \right]. \quad (2.7)$$

Up to linear and quadratic terms of t^ℓ , the prepotential takes the form [9]

$$F(t) = \frac{1}{3!} \sum_{\ell,m,n}^{h^{1,1}(X)} \kappa_{\ell mn} t^\ell t^m t^n - \frac{i}{16\pi^3} \zeta(3) \chi(X) + \widehat{F}(t), \quad (2.8)$$

where

$$\widehat{F}(t) = \frac{1}{(2\pi i)^3} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_\beta \text{Li}_3(q^\beta). \quad (2.9)$$

Here $\chi(X)$ is the Euler characteristic of X and the integers n_β are the genus zero Gromov-Witten invariants. In terms of the prepotential, the generating function (2.4) is expressed as $F_\ell(t) = \frac{\partial}{\partial t^\ell} F(t)$.

2.2 Kähler potential and $\mathcal{N} = (2, 2)$ GLSM

In this subsection, we first explain the relation between the solutions to the Picard-Fuchs equations and the Kähler potential for Calabi-Yau threefold. Then we briefly summarize the result of [11] which provides a novel prescription to compute the Kähler potential.

Both of the Kähler and the complex structure moduli spaces of Calabi-Yau d -fold are Kähler manifold equipped with the Kähler potential. According to the mirror symmetry, the quantum-corrected Kähler potential on the Kähler moduli space of a Calabi-Yau d -fold X can be identified with the Kähler potential on the complex structure moduli space of the mirror manifold X^* . For Calabi-Yau threefold, the Kähler potential can be expressed in terms of the periods as [26, 23, 24]

$$\begin{aligned} K(z, \bar{z}) &= -\log i \int_{X^*} \Omega(z) \wedge \overline{\Omega(z)} \\ &= -\log i \sum_I (\bar{T}^I \mathcal{F}_I(T) - T^I \overline{\mathcal{F}_I(T)}), \quad I = 0, 1, \dots, \ell \end{aligned} \quad (2.10)$$

up to the Kähler transformations $K(z, \bar{z}) \rightarrow K(z, \bar{z}) + f(z) + \overline{f(\bar{z})}$ where $f(z)$ is a local holomorphic function. Substituting (2.7) – (2.9), at the large radius point we obtain

$$\begin{aligned} e^{-K(z, \bar{z})} &= -\frac{i}{6} \sum_{\ell, m, n} \kappa_{\ell mn} (t^\ell - \bar{t}^\ell)(t^m - \bar{t}^m)(t^n - \bar{t}^n) + \frac{1}{4\pi^3} \zeta(3) \chi(X) \\ &\quad - \frac{i}{(2\pi i)^2} \sum_{\beta, \ell} n_\beta (\text{Li}_2(q^\beta) + \text{Li}_2(\bar{q}^\beta)) \beta_\ell (t^\ell - \bar{t}^\ell) + \frac{2i}{(2\pi i)^3} \sum_{\beta} n_\beta (\text{Li}_3(q^\beta) + \text{Li}_3(\bar{q}^\beta)), \end{aligned} \quad (2.11)$$

where we used the degrees of freedom of Kähler transformations to subtract $\log T^0(z) \overline{T^0(z)}$ from $K(z, \bar{z})$. This is a standard way to evaluate the Kähler potential by using the mirror symmetry. Even if we consider the cases in which no mirror construction is known, it is expected that the formula (2.11) still holds.

Alternatively, as conjectured in [11] and proved physically in [14], the Kähler potential on the Kähler moduli space of X can be computed from the partition function of an

$\mathcal{N} = (2, 2)$ GLSM on two-sphere S^2 via

$$e^{-K(z, \bar{z})} = Z_{\text{GLSM}}. \quad (2.12)$$

Here it is assumed that the GLSM description which flows in the IR to the non-linear sigma model on X exists. The Fayet-Iliopoulos (FI) parameters r_ℓ and the theta angles θ_ℓ of the GLSM are related to the quantum Kähler moduli $z = \{z_\ell\}$ on X via

$$z_\ell = e^{-2\pi r_\ell + i\theta_\ell}. \quad (2.13)$$

The two sphere partition function Z_{GLSM} was exactly computed in [12, 13]. Suppose the gauge group is $G \times U(1)^s$. Throughout this paper we focus on the cases of $G = U(k)$. The matter sector consists of chiral multiplets Φ_A in the irreducible representation R_A of G and we denote their charges under the $U(1)$ factors as Q_A^ℓ , $\ell = 1, \dots, s$. By performing the supersymmetric localization, the authors of [12, 13] obtained⁴

$$Z_{\text{GLSM}} = \frac{1}{|\mathcal{W}|} \sum_{\mathbf{m}=\{\mathbf{m}_i, \tilde{\mathbf{m}}_\ell\}} \int \left[\prod_{i=1}^{\text{rank}(G)} \prod_{\ell=1}^s \frac{d\sigma_i}{2\pi} \frac{d\tilde{\sigma}_\ell}{2\pi} \right] Z_{\text{class}}(\sigma, \mathbf{m}) Z_{\text{gauge}}(\{\sigma_i\}, \{\mathbf{m}_i\}) \prod_A Z_{\Phi_A}(\sigma, \mathbf{m}), \quad (2.14)$$

where $|\mathcal{W}|$ is the order of the Weyl group of G and $\sigma = \{\sigma_i, \tilde{\sigma}_\ell\}$. $\sigma_i \in \mathbb{R}^{\text{rank}(G)}$ is in the Cartan subalgebra of G and $\mathbf{m}_i \in \mathbb{Z}^{\text{rank}(G)}$ is the magnetic charge for the Cartan part of the gauge group G (GNO charge [27]). Similarly, $\tilde{\sigma}_\ell$ and $\tilde{\mathbf{m}}_\ell$ parametrize \mathbb{R}^s and \mathbb{Z}^s respectively.

The partition function consists of three pieces. The first of these is the contribution from the classical action on the localization configuration which takes the form

$$Z_{\text{class}}(\sigma, \mathbf{m}) = e^{-4\pi i(r \sum_i \sigma_i + \sum_\ell r_\ell \tilde{\sigma}_\ell) - i(\theta \sum_i \mathbf{m}_i + \sum_\ell \theta_\ell \tilde{\mathbf{m}}_\ell)}, \quad (2.15)$$

where r and θ are the FI parameter and the theta angle for central $U(1) \subset G$, respectively. $Z_{\text{gauge}}(\{\sigma_i\}, \{\mathbf{m}_i\})$ and $Z_{\Phi_A}(\sigma, \mathbf{m})$ are the one loop determinants of the vector multiplet and the chiral multiplets given by

$$Z_{\text{gauge}}(\{\sigma_i\}, \{\mathbf{m}_i\}) = \prod_{\alpha \in \Delta_+} \left(\frac{(\alpha, \mathbf{m})^2}{4} + (\alpha, \sigma)^2 \right), \quad (2.16)$$

$$Z_{\Phi_A}(\sigma, \mathbf{m}) = \prod_{w \in R_A} \frac{\Gamma(\frac{1}{2}\mathbf{q}_A - (w, i\sigma + \frac{1}{2}\mathbf{m}) - \sum_\ell Q_A^\ell (i\tilde{\sigma}_\ell + \frac{1}{2}\tilde{\mathbf{m}}_\ell))}{\Gamma(1 - \frac{1}{2}\mathbf{q}_A + (w, i\sigma - \frac{1}{2}\mathbf{m}) + \sum_\ell Q_A^\ell (i\tilde{\sigma}_\ell - \frac{1}{2}\tilde{\mathbf{m}}_\ell))}, \quad (2.17)$$

⁴Here we consider the Coulomb branch representation in which we take the coefficient of the deformation term to be infinity.

where (\cdot, \cdot) is the standard inner product, Δ_+ is the set of positive roots, w is the weight vector of the representation R_A , and \mathbf{q}_A is the $U(1)_V$ R-charge of a chiral multiplet.⁵

In the subsequent sections, we mainly focus on $d = 4$ and conjecture the exact formula of the quantum-corrected Kähler potential for Calabi-Yau fourfold. Moreover, by utilizing the proposal (2.12), we test the conjecture by computing the Gromov-Witten invariants of Calabi-Yau fourfolds and comparing them with the existing results in the literature.

3 Kähler potential for Calabi-Yau fourfolds

Now we turn to study the Kähler moduli space of a Calabi-Yau fourfold X . For $d = 4$, the non-trivial three-point A-model correlator in (2.2) is given by

$$C_{k\ell n}^{(1,1)} = \langle \mathcal{O}_k^{(1)} \mathcal{O}_\ell^{(1)} \mathcal{O}_n^{(2)} \rangle_{\mathbb{P}^1}. \quad (3.1)$$

Let us consider the observable $\mathcal{O}_n^{(2)}$ defined on the primary subspace of the cohomology $H_{\text{prim}}^{2,2}(X) \subset H^{2,2}(X)$ generated by the wedge products $J_k \wedge J_\ell$, where J_k are the elements of $H^{1,1}(X)$. Let $\{H_n\}$ be a basis of $H_{\text{prim}}^{2,2}(X)$. As in (2.4), we can define the generating function of the Gromov-Witten invariants associated with an element H_n as

$$F_n(t) = \frac{1}{2} \kappa_{k\ell n} t^k t^\ell + \widehat{F}_n(t), \quad \widehat{F}_n(t) = \frac{1}{(2\pi i)^2} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_{\beta, n} \text{Li}_2(q^\beta), \quad (3.2)$$

where $\kappa_{k\ell n} = \int_X J_k \wedge J_\ell \wedge H_n$ are the classical intersection numbers.

Here we conjecture that in the vicinity of the large radius point the quantum-corrected Kähler potential for Calabi-Yau fourfold X is given by

$$\begin{aligned} e^{-K(z, \bar{z})} = & \frac{1}{4!} \kappa_{ijkl} (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k)(t^\ell - \bar{t}^\ell) + \frac{1}{2} (\widehat{G}_{k\ell}(t) + \overline{\widehat{G}_{k\ell}(t)}) (t^k - \bar{t}^k)(t^\ell - \bar{t}^\ell) \\ & - (\widehat{H}_\ell(t) - \overline{\widehat{H}_\ell(t)}) (t^\ell - \bar{t}^\ell) + \frac{1}{2} \eta^{mn} (\widehat{F}_{mn, \ell}(t) - \overline{\widehat{F}_{mn, \ell}(t)}) (t^\ell - \bar{t}^\ell) \\ & + \frac{i}{4\pi^3} \zeta(3) C_\ell(t^\ell - \bar{t}^\ell) - \frac{1}{2} \eta^{mn} (\widehat{F}_m(t) - \overline{\widehat{F}_m(t)}) (\widehat{F}_n(t) - \overline{\widehat{F}_n(t)}) \end{aligned} \quad (3.3)$$

up to the degrees of freedom of Kähler transformation. Here we defined the generating functions associated with the elements $J_k \wedge J_\ell$ as

$$G_{k\ell}(t) = \frac{1}{2} \kappa_{ijkl} t^i t^j + \widehat{G}_{k\ell}(t), \quad \widehat{G}_{k\ell}(t) = \frac{1}{(2\pi i)^2} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_{\beta, k\ell} \text{Li}_2(q^\beta), \quad (3.4)$$

⁵Note that, as discussed in [13], the theory on S^2 breaks the classical $U(1)_A$ R-symmetry of the original $\mathcal{N} = (2, 2)$ GLSM in flat space. It was also emphasized that unitarity constraints the R-charges to be non-negative. If we consider non-compact Calabi-Yau variety as target space, all the R-charges of chiral fields of GLSM should be set to zero [28]. We will encounter these cases in Section 5.

where the coefficients $\kappa_{ijkl} = \int_X J_i \wedge J_j \wedge J_k \wedge J_\ell$ are the classical quadruple intersection numbers. Note that such a generating function $G_{k\ell}(t)$ can be obtained by a linear combination of $F_n(t)$.⁶ We have also defined

$$\widehat{H}_\ell(t) = \int_{i\infty}^{t^\ell} \widehat{G}_{\ell\ell}(t) d\tilde{t}^\ell + 2 \sum_{k \neq \ell} \int_{i\infty}^{t^k} \widehat{G}_{k\ell}(t) d\tilde{t}^k \Big|_{t^\ell = i\infty}, \quad (3.5)$$

$$\widehat{F}_{mn;\ell}(t) = \int_{i\infty}^{t^\ell} \tilde{\partial}_\ell \widehat{F}_m(t) \tilde{\partial}_\ell \widehat{F}_n(t) d\tilde{t}^\ell, \quad (3.6)$$

$$C_\ell = \int_X c_3(X) \wedge J_\ell, \quad (3.7)$$

and $\eta^{mn} = \eta_{mn}^{-1}$ is the inverse matrix of the intersection matrix $\eta_{mn} = \int_X H_m \wedge H_n$ on $H_{\text{prim}}^{2,2}(X)$. $c_3(X)$ is the third Chern class of X . In (3.5) and (3.6), we abbreviated the arguments of the integrands as $t = \{t^1, \dots, t^{\ell-1}, \tilde{t}^\ell, t^{\ell+1}, \dots, t^{h^{1,1}}\}$. Our expression (3.3) is exactly the four dimensional extension of (2.11) which has not been fully understood.

Let us revisit our conjecture (3.3) from the viewpoint of the B-model. Suppose we consider a Calabi-Yau fourfold X whose mirror construction is known. As explained in Section 2, on the B-model side, the solutions to Picard-Fuchs equations give the periods of a holomorphic d -form $\Omega(z)$ on the mirror manifold. Here we assume that the periods take the following forms

$$\begin{aligned} \Pi^{(0)}(z) &= T^0(z), \quad \frac{\Pi_\ell^{(1)}(z)}{2\pi i T^0(z)} = t^\ell, \quad \frac{\Pi_n^{(2)}(z)}{(2\pi i)^2 T^0(z)} = \frac{1}{2} \kappa_{\ell mn} t^\ell t^m + \widehat{F}_n(t), \\ \frac{\Pi_\ell^{(3)}(z)}{(2\pi i)^3 T^0(z)} &= \frac{1}{3!} \kappa_{ijkl} t^i t^j t^k + \widehat{G}_{k\ell}(t) t^k - \widehat{H}_\ell(t) + \frac{1}{2} \eta^{mn} \widehat{F}_{mn;\ell}(t) + \frac{i}{8\pi^3} \zeta(3) C_\ell, \\ \frac{\Pi^{(4)}(z)}{(2\pi i)^4 T^0(z)} &= \frac{1}{4!} \kappa_{ijkl} t^i t^j t^k t^\ell + \frac{1}{2} \widehat{G}_{k\ell}(t) t^k t^\ell - \widehat{H}_\ell(t) t^\ell + \frac{1}{2} \eta^{mn} \widehat{F}_{mn;\ell}(t) t^\ell \\ &\quad + \frac{i}{8\pi^3} \zeta(3) C_\ell t^\ell - \frac{1}{2} \eta^{mn} \widehat{F}_m(t) \widehat{F}_n(t). \end{aligned} \quad (3.8)$$

in the vicinity of the large complex structure point.⁷ Plugging these periods into a four dimensional analogue of (2.10) given by

$$e^{-K(z, \bar{z})} = \left[\Pi^{(0)}(z) \overline{\Pi^{(4)}(z)} + \sum_\ell \Pi_\ell^{(1)}(z) \overline{\Pi_\ell^{(3)}(z)} + c.c. \right] + \eta^{mn} \Pi_m^{(2)}(z) \overline{\Pi_n^{(2)}(z)}, \quad (3.9)$$

we can obtain the conjectural formula (3.3).

⁶In general we have multiple elements of $H_{\text{prim}}^{2,2}(X)$ within the wedge product $J_k \wedge J_\ell$.

⁷This should be obtained by the Frobenius method as in the case of the complete intersection Calabi-Yau threefolds in projective spaces [8, 9] (see also [29] which proposed an alternative method for Calabi-Yau fourfolds by using the analytic continuation to a conifold point and a monodromy analysis). In the context of open mirror symmetry, the relative periods of Calabi-Yau threefolds with branes were studied in [30, 31] and it was also mentioned that these are related to the periods of Calabi-Yau fourfolds without branes. It would be interesting to provide further details of the relationship to our conjecture.

4 Examples

According to the relation (2.12) proposed in [11], it should be possible to verify our conjecture (3.3) about the exact Kähler potential for Calabi-Yau fourfold by comparing with the $\mathcal{N} = (2, 2)$ GLSM partition function. In this section, through the use of (3.3), we extract the topological invariants such as the genus zero Gromov-Witten invariants from the GLSM partition function, and show that our conjecture (3.3) is consistent with the mirror symmetry predictions.

Here we explain a prescription to extract the topological data from GLSM calculation. As we mentioned in Section 2.2, the quantum Kähler moduli z_ℓ on a Calabi-Yau manifold X are related to the FI parameters and the theta angles of the corresponding GLSM. In order to evaluate the Kähler potential in the vicinity of the large radius point $z_\ell = 0$, we need to find the flat coordinates t^ℓ which give the classical Kähler moduli. As explained in [11], the flat coordinates can be determined by the following procedure. First we perform the contour integration of the two sphere partition function Z_{GLSM} around the large radius point. As indicated in (3.3), the coefficients of $\frac{1}{4!} \log \bar{z}_i \log \bar{z}_j \log \bar{z}_k \log \bar{z}_\ell$ in Z_{GLSM} should be the classical intersection numbers κ_{ijkl} . Note that we need to use the degrees of freedom of Kähler transformation $K(z, \bar{z}) \rightarrow K(z, \bar{z}) + f(z) + \overline{f(\bar{z})}$ in order to obtain appropriate intersection numbers. This corresponds to the normalization of the partition function.⁸ This is similar to the situation in the Calabi-Yau threefold in [11]. After performing the Kähler transformation, from the coefficients of $\log \bar{z}_i \log \bar{z}_j \log \bar{z}_k$ which should be identified with $\frac{1}{3!(2\pi i)^3} \kappa_{ijk\ell} t^\ell$, we can determine the flat coordinates which take the form

$$2\pi i t^\ell = \log z_\ell + 2\pi i t_{(0)}^\ell + \Delta_\ell(z). \quad (4.1)$$

Here $\Delta_\ell(z)$ are holomorphic functions and $0 \leq t_{(0)}^\ell < 1$ are constants fixed by requiring the positivity of Gromov-Witten invariants. By inverting (4.1), we can express the z_ℓ in terms of t^ℓ as $z_\ell = e^{-2\pi i t_{(0)}^\ell} q_\ell + \mathcal{O}(q^2)$, where $q_\ell = e^{2\pi i t^\ell}$. Reading off the coefficients of $\log \bar{z}_k \log \bar{z}_\ell$ in the q -expansion, we obtain the generating functions (3.4). From the coefficients of $\log \bar{z}_\ell$, we can also find all the remaining generating functions (3.2) associated with $H_{\text{prim}}^{2,2}(X)$.

4.1 Sextic fourfold: $X_6 \subset \mathbb{P}^5$

As a simplest example of the compact Calabi-Yau fourfold, let us consider the Fermat sextic fourfold $X_6 \subset \mathbb{P}^5$ defined by a degree six hypersurface in \mathbb{P}^5 [6] (see also [29]).

⁸As we will see later, $f(z)$ coincides with the logarithm of a solution $\Pi^{(0)}(z) = T^0(z)$ to the corresponding Picard-Fuchs equations.

Field	U(1)	$U(1)_V$
Φ_i	+1	$2\mathfrak{q}$
P	-6	$2 - 12\mathfrak{q}$

Table 1: Matter content of the abelian GLSM for the sextic fourfold in \mathbb{P}^5 . Here $i = 1, \dots, 6$. We set the R-charges in such a way that the total R-charge of superpotential becomes 2. The positivity of the R-charges implies $0 < \mathfrak{q} < \frac{1}{6}$.

There is one Kähler modulus associated with the radius of \mathbb{P}^5 . First we determine the topological data of this manifold. Using the Kähler form J of \mathbb{P}^5 , the classical quadruple intersection number is computed as $\kappa = \int_{X_6} J^4 = \int_{\mathbb{P}^5} 6J^5 = 6$. From the total Chern class $c(X_6) = \frac{(1+J)^6}{1+6J}$, we can also see that $\int_{X_6} c_3(X_6) \wedge J = -70 \int_{X_6} J^3 \wedge J = -420$.

The sextic fourfold has an abelian $\mathcal{N} = (2, 2)$ GLSM description with matter content shown in Table 1. This model has a superpotential $W = PW_6(\Phi)$ where $W_6(\Phi)$ is a homogeneous degree six polynomial of Φ_i . The GLSM has a phase transition which occurs as the FI parameter r is varied. Here we consider the Calabi-Yau phase $r \gg 0$.

Using the formulas (2.14) – (2.17), we can write the exact GLSM partition function for the sextic fourfold as

$$Z_{\text{GLSM}} = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \frac{\Gamma(\mathfrak{q} - i\sigma - \frac{1}{2}m)^6}{\Gamma(1 - \mathfrak{q} + i\sigma - \frac{1}{2}m)^6} \frac{\Gamma(1 - 6\mathfrak{q} + 6i\sigma + 3m)}{\Gamma(6\mathfrak{q} - 6i\sigma + 3m)}. \quad (4.2)$$

This can be evaluated in the same way as performed in [11] and we obtain

$$Z_{\text{GLSM}} = (z\bar{z})^{\mathfrak{q}} \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{-\epsilon} \frac{\pi^5 \sin(6\pi\epsilon)}{\sin^6(\pi\epsilon)} \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 6k - 6\epsilon)}{\Gamma(1 + k - \epsilon)^6} \right|^2, \quad (4.3)$$

where $z = e^{-2\pi r + i\theta}$. Note that the complex conjugation does not act on ϵ .

As explained above, we first look at the coefficient of $\log^4 \bar{z}$. The result is given by

$$\frac{6}{4!} (z\bar{z})^{\mathfrak{q}} T^0(z) \overline{T^0(z)}, \quad T^0(z) = \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 6k)}{\Gamma(1 + k)^6}. \quad (4.4)$$

We can show that $T^0(z)$ is a kernel of the Picard-Fuchs operator associated with the mirror manifold of the sextic fourfold given by

$$\mathcal{D} = \Theta^5 - 6z \prod_{k=1}^5 (6\Theta + k), \quad \Theta = z \frac{\partial}{\partial z}. \quad (4.5)$$

After dividing the partition function Z_{GLSM} by $(z\bar{z})^{\mathfrak{q}} (2\pi i)^4 T^0 \overline{T^0}$, we turn to determine the flat coordinate. From the coefficient of $\log^3 \bar{z}$, we can read off the flat coordinate

$$2\pi i t = \log z + 2\pi i t_{(0)} + \frac{6}{T^0(z)} \sum_{k=1}^{\infty} \frac{(6k)!}{(k!)^6} z^k [\Psi(1 + 6k) - \Psi(1 + k)], \quad (4.6)$$

d	n_d
1	60480
2	440884080
3	6255156277440
4	117715791990353760

Table 2: Gromov-Witten invariants for $X_6 \subset \mathbb{P}^5$.

Field	U(1)	$U(1)_V$
Φ_i	+1	$2\mathfrak{q}$
P_a	$-d_a$	$2 - 2d_a\mathfrak{q}$

Table 3: Matter content of the abelian GLSMs for the complete intersection Calabi-Yau fourfolds (4.8). Here $i = 1, \dots, n$ and $a = 1, \dots, r$.

where $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function and $t_{(0)}$ is a constant. By inverting this flat coordinate and substituting it into the partition function (4.3), we can extract the Gromov-Witten invariants. With the choice of $t_{(0)} = 0$, the generating function of these invariants (3.4) associated with J^2 is obtained as

$$G(t) = \frac{6}{2}t^2 + \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} n_d \text{Li}_2(q^d), \quad q = e^{2\pi i t}, \quad (4.7)$$

where the Gromov-Witten invariants n_d are summarised in Table 2. The result is in perfect agreement with the mirror symmetry predictions of [3, 6]. We can also find that the explicit form of Z_{GLSM} matches with our conjecture (3.3) via (2.12).

We can also consider general complete intersection Calabi-Yau manifolds $X_{d_1, \dots, d_r} \subset \mathbb{P}^n$ defined by r hypersurfaces with the degrees (d_1, \dots, d_r) in the projective space \mathbb{P}^n . The complex dimension of these manifolds is determined by $n-r$, and the Calabi-Yau condition is satisfied when $d_1 + \dots + d_r = n+1$. Then, including the above example, we can construct seven such fourfolds

$$X_6 \subset \mathbb{P}^5, \quad X_{2,5} \subset \mathbb{P}^6, \quad X_{3,4} \subset \mathbb{P}^6, \quad X_{2,2,4} \subset \mathbb{P}^7, \quad X_{2,3,3} \subset \mathbb{P}^7, \quad X_{2,2,2,3} \subset \mathbb{P}^8, \quad X_{2,2,2,2,2} \subset \mathbb{P}^9. \quad (4.8)$$

All these examples have one Kähler form J . The GLSM for each fourfold has the matter content shown in Table 3. There are r superpotentials $W_a = P_a W_{d_a}(\Phi)$, $a = 1, \dots, r$ where $W_{d_a}(\Phi)$ is a homogeneous degree d_a polynomial of Φ_i . In Table 4 we summarize our results for Gromov-Witten invariants n_d associated with J^2 calculated in the same

	$X_{2,5} \subset \mathbb{P}^6$ ($\kappa = 10$)	$X_{3,4} \subset \mathbb{P}^6$ ($\kappa = 12$)	$X_{2,2,4} \subset \mathbb{P}^7$ ($\kappa = 16$)
n_1	24500	16128	11776
n_2	48263250	17510976	7677952
n_3	181688069500	36449586432	9408504320
n_4	905026660335000	100346754888576	15215566524416
n_5	5268718476406938000	322836001522723584	28735332663693824
	$X_{2,3,3} \subset \mathbb{P}^7$ ($\kappa = 18$)	$X_{2,2,2,3} \subset \mathbb{P}^8$ ($\kappa = 24$)	$X_{2,2,2,2,2} \subset \mathbb{P}^9$ ($\kappa = 32$)
n_1	9396	6912	5120
n_2	4347594	1919808	852480
n_3	3794687028	988602624	259476480
n_4	4368985908840	669909315456	103646279680
n_5	5873711971817268	529707745490688	48276836019200

Table 4: Gromov-Witten invariants for complete intersection Calabi-Yau fourfolds in the projective space. Here κ is the classical quadruple intersection number.

way as the sextic example. We have also checked our conjecture (3.3) holds in these examples.

4.2 Quintic fibration over \mathbb{P}^1 : $X_{2,5} \subset \mathbb{P}^1 \times \mathbb{P}^4$

As an example with two Kähler moduli, we consider a quintic fibration over \mathbb{P}^1 expressed by $X_{2,5} \subset \mathbb{P}^1 \times \mathbb{P}^4$ which is defined as a Calabi-Yau hypersurface with degree two and degree five for the coordinates of \mathbb{P}^1 and \mathbb{P}^4 , respectively [6]. Denoting the Kähler forms on \mathbb{P}^1 and \mathbb{P}^4 by J_1 and J_2 , the nonzero classical quadruple intersection numbers are computed as $\kappa_{1222} = \int_{X_{2,5}} J_1 \wedge J_2^3 = \int_{\mathbb{P}^1 \times \mathbb{P}^4} J_1 \wedge J_2^3 \wedge (2J_1 + 5J_2) = 5$ and $\kappa_{2222} = \int_{X_{2,5}} J_2^4 = \int_{\mathbb{P}^1 \times \mathbb{P}^4} J_2^4 \wedge (2J_1 + 5J_2) = 2$. The total Chern class of this manifold is $c(X_{2,5}) = \frac{(1+J_1)^2(1+J_2)^5}{1+2J_1+5J_2}$ and thus we see that $\int_{X_{2,5}} c_3(X_{2,5}) \wedge J_1 = -200$ and $\int_{X_{2,5}} c_3(X_{2,5}) \wedge J_2 = -330$.

Corresponding $\mathcal{N} = (2, 2)$ GLSM has two $U(1)$ gauge groups with matter fields summarised in Table 5. These fields interact through a superpotential $W = PW_{2,5}(\Phi_1, \Phi_2)$, where $W_{2,5}(\Phi_1, \Phi_2)$ is a homogeneous degree two and degree five polynomial of Φ_{1,i_1} and Φ_{2,i_2} , respectively. There are two FI parameters associated with $U(1)_1 \times U(1)_2$ gauge symmetry and we consider the Calabi-Yau phase $r_1, r_2 \gg 0$.

According to the localization formulas (2.14) – (2.17), the exact partition function for

Field	$U(1)_1$	$U(1)_2$	$U(1)_V$
Φ_{1,i_1}	+1	0	$2\mathbf{q}_1$
Φ_{2,i_2}	0	+1	$2\mathbf{q}_2$
P	-2	-5	$2 - 4\mathbf{q}_1 - 10\mathbf{q}_2$

Table 5: Matter content of the $U(1)_1 \times U(1)_2$ GLSM for the Calabi-Yau $X_{2,5} \subset \mathbb{P}^1 \times \mathbb{P}^4$. Here $i_1 = 1, 2$ and $i_2 = 1, \dots, 5$. The R-charges are assigned so that the total R-charge of superpotential is 2. The positivity of R-charges requires $\mathbf{q}_1 > 0$, $\mathbf{q}_2 > 0$, and $2\mathbf{q}_1 + 5\mathbf{q}_2 < 1$.

the quintic fibration $X_{2,5}$ is given by

$$\begin{aligned}
Z_{\text{GLSM}} = & \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i(\theta_1 m_1 + \theta_2 m_2)} \int_{-\infty}^{\infty} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} e^{-4\pi i(r_1 \sigma_1 + r_2 \sigma_2)} \\
& \times \frac{\Gamma(\mathbf{q}_1 - i\sigma_1 - \frac{1}{2}m_1)^2}{\Gamma(1 - \mathbf{q}_1 + i\sigma_1 - \frac{1}{2}m_1)^2} \frac{\Gamma(\mathbf{q}_2 - i\sigma_2 - \frac{1}{2}m_2)^5}{\Gamma(1 - \mathbf{q}_2 + i\sigma_2 - \frac{1}{2}m_2)^5} \\
& \times \frac{\Gamma(1 - 2\mathbf{q}_1 - 5\mathbf{q}_2 + i(2\sigma_1 + 5\sigma_2) + (m_1 + \frac{5}{2}m_2))}{\Gamma(2\mathbf{q}_1 + 5\mathbf{q}_2 - i(2\sigma_1 + 5\sigma_2) + (m_1 + \frac{5}{2}m_2))}. \tag{4.9}
\end{aligned}$$

In a similar manner to the sextic fourfold, we can evaluate this into the form

$$\begin{aligned}
Z_{\text{GLSM}} = & (z_1 \bar{z}_1)^{\mathbf{q}_1} (z_2 \bar{z}_2)^{\mathbf{q}_2} \oint \frac{d\epsilon_1}{2\pi i} \frac{d\epsilon_2}{2\pi i} (z_1 \bar{z}_1)^{-\epsilon_1} (z_2 \bar{z}_2)^{-\epsilon_2} \frac{\pi^6 \sin \pi(2\epsilon_1 + 5\epsilon_2)}{\sin^2(\pi\epsilon_1) \sin^5(\pi\epsilon_2)} \\
& \times \left| \sum_{k_1, k_2=0}^{\infty} z_1^{k_1} (-z_2)^{k_2} \frac{\Gamma(1 + (2k_1 + 5k_2) - (2\epsilon_1 + 5\epsilon_2))}{\Gamma(1 + k_1 - \epsilon_1)^2 \Gamma(1 + k_2 - \epsilon_2)^5} \right|^2, \tag{4.10}
\end{aligned}$$

where we defined $z_\ell = e^{-2\pi r_\ell + i\theta_\ell}$ and the complex conjugation does not act on $\epsilon_{1,2}$.

Let us consider $\log^4 \bar{z}_2$ term. The coefficient is given by

$$\frac{2}{4!} (z_1 \bar{z}_1)^{\mathbf{q}_1} (z_2 \bar{z}_2)^{\mathbf{q}_2} T^0(z_1, z_2) \overline{T^0(z_1, z_2)}, \tag{4.11}$$

where

$$T^0(z_1, z_2) = \sum_{k_1, k_2=0}^{\infty} z_1^{k_1} (-z_2)^{k_2} \frac{\Gamma(1 + 2k_1 + 5k_2)}{\Gamma(1 + k_1)^2 \Gamma(1 + k_2)^5}. \tag{4.12}$$

We see that $T^0(z_1, -z_2)$ is annihilated by the Picard-Fuchs operators associated with $X_{2,5}$ defined by

$$\begin{aligned}
\mathcal{D}_1 = & \Theta_1^2 - z_1 \prod_{k=1}^2 (2\Theta_1 + 5\Theta_2 + k), \\
\mathcal{D}_2 = & (2\Theta_1 - 5\Theta_2)\Theta_2^3 - 4z_1(2\Theta_1 + 5\Theta_2 + 1)\Theta_2^3 + 25z_2 \prod_{k=1}^4 (2\Theta_1 + 5\Theta_2 + k), \tag{4.13}
\end{aligned}$$

$n_{d_1, d_2, 12}$	$d_1 = 0$	1	2	3	4
$d_2 = 0$		125	0	0	0
1	2875	195875	1248250	1799250	662875
2	1218500	369229625	10980854250	101591346500	384568351000
$n_{d_1, d_2, 22}$	$d_1 = 0$	1	2	3	4
$d_2 = 0$		0	0	0	0
1	9950	171750	609500	609500	171750
2	5487450	533197250	9651689750	63917722000	188112166000

Table 6: Gromov-Witten invariants for $X_{2,5} \subset \mathbb{P}^1 \times \mathbb{P}^4$. Note that the assignments of d_1 and d_2 are exchanged in [6].

where $\Theta_\ell = z_\ell \frac{\partial}{\partial z_\ell}$.

Normalizing the partition function Z_{GLSM} by $(z_1 \bar{z}_1)^{q_1} (z_2 \bar{z}_2)^{q_2} (2\pi i)^4 T^0 \overline{T^0}$, we can read off the flat coordinates from the coefficients of $\log^3 \bar{z}_2$ and $\log \bar{z}_1 \log^2 \bar{z}_2$ as

$$2\pi i t^1 = \log z_1 + 2\pi i t_{(0)}^1 + \frac{2}{T^0} \sum_{k_1, k_2=0}^{\infty} \frac{(2k_1 + 5k_2)!}{(k_1!)^2 (k_2!)^5} z_1^{k_1} (-z_2)^{k_2} [\Psi(1 + 2k_1 + 5k_2) - \Psi(1 + k_1)], \quad (4.14)$$

$$2\pi i t^2 = \log z_2 + 2\pi i t_{(0)}^2 + \frac{5}{T^0} \sum_{k_1, k_2=0}^{\infty} \frac{(2k_1 + 5k_2)!}{(k_1!)^2 (k_2!)^5} z_1^{k_1} (-z_2)^{k_2} [\Psi(1 + 2k_1 + 5k_2) - \Psi(1 + k_2)], \quad (4.15)$$

where $0 \leq t_{(0)}^1, t_{(0)}^2 < 1$ are constants. Inverting these flat coordinates and taking $t_{(0)}^1 = 0$, $t_{(0)}^2 = \frac{1}{2}$, we can finally obtain the generating functions (3.4) as

$$G_{12}(t_1, t_2) = \frac{5}{2} t_2^2 + \frac{1}{(2\pi i)^2} \sum_{\substack{d_1, d_2=0 \\ (d_1, d_2) \neq (0,0)}}^{\infty} n_{d_1, d_2, 12} \text{Li}_2(q_1^{d_1} q_2^{d_2}), \quad (4.16)$$

$$G_{22}(t_1, t_2) = \frac{2}{2} t_2^2 + \frac{1}{(2\pi i)^2} \sum_{\substack{d_1, d_2=0 \\ (d_1, d_2) \neq (0,0)}}^{\infty} n_{d_1, d_2, 22} \text{Li}_2(q_1^{d_1} q_2^{d_2}), \quad (4.17)$$

where the Gromov-Witten invariants are listed in Table 6 and we see that these integer invariants completely agree with the result of [6]. We have also checked that the exact partition function coincides with (3.3) up to q_1^4 and q_2^4 .

Field	U(1)	U(1)	$U(1)_V$
Φ_i	+1	0	$2\mathbf{q}_\phi$
P_a	-1	+1	$2 - 2\mathbf{q}_x - 2\mathbf{q}_\phi$
X_a	0	-1	$2\mathbf{q}_x$

Table 7: Matter content of the abelian PAX GLSM for the determinantal sextic fourfold in \mathbb{P}^5 . Here $i = 1, \dots, 6$ and $a = 1, \dots, 6$.

4.3 Resolved determinantal sextic in \mathbb{P}^5 : $X_A \subset \mathbb{P}^5 \times \mathbb{P}^5$

Here we consider the linear determinantal Calabi-Yau fourfold defined by

$$Z(A, 5) = \{\phi \in \mathbb{P}^5 \mid \text{rank}(A^i \phi_i) \leq 5\}, \quad (4.18)$$

where the A^i are six 5×5 constant matrices and the ϕ_i are the homogeneous coordinates of \mathbb{P}^5 . A GLSM construction for determinantal manifolds was studied in [32]. Following their prescription, we analyze the determinantal sextic fourfold (4.18) using the $U(1) \times U(1)$ “PAX” model with matter content shown in Table 7. These matter multiplets interact through a superpotential $W = \text{tr}(PA^i \Phi_i X)$. This model has three distinct geometric phases and the “ X_A phase” [32, 11]

$$X_A = \{(\phi, x) \in \mathbb{P}^5 \times \mathbb{P}^5 \mid (A^i \phi_i)x = 0\} \quad (4.19)$$

gives a resolution of the determinantal variety (4.18). Here we denote the Kähler form on the first (base) \mathbb{P}^5 by J_1 , and the hyperplane class of the second (fiber) \mathbb{P}^5 by J_2 . Then the classical quadruple intersection numbers are computed as $\kappa_{1111} = \kappa_{2222} = 6$, $\kappa_{1112} = \kappa_{1222} = 15$, and $\kappa_{1122} = 20$ from the top Chern class of a rank six normal bundle \mathcal{X} whose total Chern class is given by $c(\mathcal{X}) = (1 + J_1 + J_2)^6$. The total Chern class of X_A is given by $c(X_A) = \frac{(1+J_1)^6(1+J_2)^6}{c(\mathcal{X})}$, and the other topological invariants are also obtained as $\int_{X_A} c_3(X_A) \wedge J_1 = \int_{X_A} c_3(X_A) \wedge J_2 = -210$ [32].

The partition function of this PAX GLSM is given by

$$\begin{aligned}
Z_{\text{GLSM}} = & \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i(\theta_1 m_1 + \theta_2 m_2)} \int_{-\infty}^{\infty} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} e^{-4\pi i(r_1 \sigma_1 + r_2 \sigma_2)} \\
& \times \frac{\Gamma(\mathbf{q}_\phi - i\sigma_1 - \frac{1}{2}m_1)^6}{\Gamma(1 - \mathbf{q}_\phi + i\sigma_1 - \frac{1}{2}m_1)^6} \frac{\Gamma(\mathbf{q}_x + i\sigma_2 + \frac{1}{2}m_2)^6}{\Gamma(1 - \mathbf{q}_x - i\sigma_2 + \frac{1}{2}m_2)^6} \\
& \times \frac{\Gamma(1 - \mathbf{q}_\phi - \mathbf{q}_x + i(\sigma_1 - \sigma_2) + \frac{1}{2}(m_1 - m_2))^6}{\Gamma(\mathbf{q}_\phi + \mathbf{q}_x - i(\sigma_1 - \sigma_2) + \frac{1}{2}(m_1 - m_2))^6}. \quad (4.20)
\end{aligned}$$

Here we take a phase $r_1 \gg 0$, $r_2 \ll 0$ corresponding to the X_A phase, where r_1 and r_2 correspond to J_1 and J_2 , respectively. As in the previous examples, the partition function can be evaluated as

$$Z_{\text{GLSM}} = (z_1 \bar{z}_1)^{q_\phi} (z_2 \bar{z}_2)^{q_x} \oint \frac{d\epsilon_1}{2\pi i} \frac{d\epsilon_2}{2\pi i} (z_1 \bar{z}_1)^{-\epsilon_1} (z_2 \bar{z}_2)^{-\epsilon_2} \frac{\pi^6 \sin^6 \pi(\epsilon_1 + \epsilon_2)}{\sin^6(\pi\epsilon_1) \sin^6(\pi\epsilon_2)} \\ \times \left| \sum_{k_1, k_2=0}^{\infty} z_1^{k_1} z_2^{k_2} \frac{\Gamma(1+k_1+k_2-\epsilon_1-\epsilon_2)^6}{\Gamma(1+k_1-\epsilon_1)^6 \Gamma(1+k_2-\epsilon_2)^6} \right|^2, \quad (4.21)$$

where $z_1 = e^{-2\pi r_1 + i\theta_1}$ and $z_2 = e^{2\pi r_2 - i\theta_2}$.⁹

The coefficient of $\log^4 \bar{z}_1$ term is

$$\frac{6}{4!} (z_1 \bar{z}_1)^{q_\phi} (z_2 \bar{z}_2)^{q_x} T^0(z_1, z_2) \overline{T^0(z_1, z_2)}, \quad (4.22)$$

where

$$T^0(z_1, z_2) = \sum_{k_1, k_2=0}^{\infty} z_1^{k_1} z_2^{k_2} \frac{\Gamma(1+k_1+k_2)^6}{\Gamma(1+k_1)^6 \Gamma(1+k_2)^6}. \quad (4.23)$$

As before, after normalizing the partition function by $(z_1 \bar{z}_1)^{q_\phi} (z_2 \bar{z}_2)^{q_x} (2\pi i)^4 T^0 \overline{T^0}$, from the coefficients of $\log \bar{z}_1 \log^2 \bar{z}_2$ and $\log^3 \bar{z}_2$, the flat coordinates are determined as

$$2\pi i t^1 = \log z_1 + 2\pi i t_{(0)}^1 + \Delta(z_1, z_2), \quad 2\pi i t^2 = \log z_2 + 2\pi i t_{(0)}^2 + \Delta(z_2, z_1), \\ \Delta(z_1, z_2) = \frac{6}{T^0} \sum_{k_1, k_2=0}^{\infty} \frac{((k_1+k_2)!)^6}{(k_1!)^6 (k_2!)^6} z_1^{k_1} z_2^{k_2} [\Psi(1+k_1+k_2) - \Psi(1+k_1)], \quad (4.24)$$

where $0 \leq t_{(0)}^1, t_{(0)}^2 < 1$ are constants. By taking $t_{(0)}^1 = t_{(0)}^2 = 0$, we obtain three independent generating functions (3.4) associated with J_1^2 , $J_1 \wedge J_2$, and J_2^2 as

$$G_{11}(t_1, t_2) = \frac{6}{2} t_1^2 + 15 t_1 t_2 + \frac{20}{2} t_2^2 + \frac{1}{(2\pi i)^2} \sum_{\substack{d_1, d_2=0 \\ (d_1, d_2) \neq (0,0)}}^{\infty} n_{d_1, d_2, 11} \text{Li}_2(q_1^{d_1} q_2^{d_2}), \quad (4.25)$$

$$G_{12}(t_1, t_2) = \frac{15}{2} t_1^2 + 20 t_1 t_2 + \frac{15}{2} t_2^2 + \frac{1}{(2\pi i)^2} \sum_{\substack{d_1, d_2=0 \\ (d_1, d_2) \neq (0,0)}}^{\infty} n_{d_1, d_2, 12} \text{Li}_2(q_1^{d_1} q_2^{d_2}), \quad (4.26)$$

and $G_{22}(t_1, t_2) = G_{11}(t_2, t_1)$, where the Gromov-Witten invariants are listed in Table 8. We have also checked the conjecture (3.3) is in agreement with the GLSM partition function.

⁹Also in the “ X_{A^*} phase” [32] corresponding to $r_1 + r_2 \gg 0$, $r_2 \gg 0$, we see that the GLSM partition function takes the same form (4.21) with $z_1 = e^{-2\pi(r_1+r_2)+i(\theta_1+\theta_2)}$ and $z_2 = e^{-2\pi r_2 + i\theta_2}$.

$n_{d_1, d_2, 11}$	$d_1 = 0$	1	2	3	4
$d_2 = 0$		210	0	0	0
1	0	5670	59430	100170	34650
2	0	24360	2579640	47382930	264433680
3	0	24360	28015260	2324403900	55841697870
4	0	5670	107096220	38404166850	2848564316640
$n_{d_1, d_2, 12}$	$d_1 = 0$	1	2	3	4
$d_2 = 0$		105	0	0	0
1	105	6930	50715	71085	21420
2	0	50715	3166800	45928155	221593050
3	0	71085	45928155	2851172100	57546197940
4	0	21420	221593050	57546197940	3492450469200

Table 8: Gromov-Witten invariants for X_A .

By using the holomorphic function (4.23) which should be identified with the fundamental period of the mirror manifold of X_A , we can find the Picard-Fuchs operators

$$\begin{aligned} \mathcal{D}_1 = & (\Theta_1 + \Theta_2)(5\Theta_1^2 - 8\Theta_1\Theta_2 + 5\Theta_2^2) - z_1(5\Theta_1^3 + 27\Theta_1^2\Theta_2 + 54\Theta_1\Theta_2^2 + 42\Theta_2^3 + 15\Theta_1^2 \\ & + 54\Theta_1\Theta_2 + 54\Theta_2^2 + 15\Theta_1 + 27\Theta_2 + 5) - z_2(42\Theta_1^3 + 54\Theta_1^2\Theta_2 + 27\Theta_1\Theta_2^2 + 5\Theta_2^3 \\ & + 54\Theta_1^2 + 54\Theta_1\Theta_2 + 15\Theta_2^2 + 27\Theta_1 + 15\Theta_2 + 5), \end{aligned} \quad (4.27)$$

$$\begin{aligned} \mathcal{D}_2 = & (\Theta_1 - \Theta_2)(5\Theta_1^2 - 4\Theta_1\Theta_2 + 5\Theta_2^2) - z_1(\Theta_1 + \Theta_2 + 1)(5\Theta_1^2 + 16\Theta_1\Theta_2 + 14\Theta_2^2 + 10\Theta_1 \\ & + 16\Theta_2 + 5) + z_2(\Theta_1 + \Theta_2 + 1)(14\Theta_1^2 + 16\Theta_1\Theta_2 + 5\Theta_2^2 + 16\Theta_1 + 10\Theta_2 + 5), \end{aligned} \quad (4.28)$$

where $\Theta_\ell = z_\ell \frac{\partial}{\partial z_\ell}$. These operators can be also derived by the method of [9, 33, 34]. Following them, first we can obtain the Picard-Fuchs operators $\widehat{\mathcal{D}}_1 = \Theta_1^6 - z_1(\Theta_1 + \Theta_2 + 1)^6$ and $\widehat{\mathcal{D}}_2 = \Theta_2^6 - z_2(\Theta_1 + \Theta_2 + 1)^6$ from the charge assignment in Table 7. Then the above operators (4.27) and (4.28) are derived from the irreducible factorizations as

$$280\widehat{\mathcal{D}}_1 = (\Theta_1 + \Theta_2)(14\Theta_1^2 + 16\Theta_1\Theta_2 + 5\Theta_2^2)\mathcal{D}_1 + (42\Theta_1^3 + 54\Theta_1^2\Theta_2 + 27\Theta_1\Theta_2^2 + 5\Theta_2^3)\mathcal{D}_2, \quad (4.29)$$

$$280\widehat{\mathcal{D}}_2 = (\Theta_1 + \Theta_2)(5\Theta_1^2 + 16\Theta_1\Theta_2 + 14\Theta_2^2)\mathcal{D}_1 - (5\Theta_1^3 + 27\Theta_1^2\Theta_2 + 54\Theta_1\Theta_2^2 + 42\Theta_2^3)\mathcal{D}_2. \quad (4.30)$$

Using the above results (4.23) – (4.26), we have checked that the conjectural forms (3.8) are annihilated by the Picard-Fuchs operators (4.27) and (4.28) up to z_1^4 and z_2^4 .

Field	$U(2)$	$U(1)_V$
Φ^i	$\mathbf{2}_{+1}$	$2\mathfrak{q}$
P_a	$\mathbf{1}_{-2}$	$2 - 4\mathfrak{q}$

Table 9: Matter content of the $U(2)$ GLSM describing the complete intersection Calabi-Yau in Grassmannian $X_{1^8} \subset G(2, 8)$. Here $i, a = 1, \dots, 8$. The subscript denotes the charge under the central $U(1) \subset U(2)$. R-charges are assigned so that the total R-charge of superpotential is 2, and by the positivity constraint, $0 < \mathfrak{q} < \frac{1}{2}$.

4.4 Complete intersection in Grassmannian: $X_{1^8} \subset G(2, 8)$

As an example with non-abelian GLSM description, we consider the Grassmannian Calabi-Yau fourfold $X_{1^8} \subset G(2, 8)$ defined by the complete intersection of eight hyperplanes with degree one in the Grassmannian $G(2, 8)$. First we compute the classical topological invariants of this manifold. We denote the class of a hyperplane section as $\sigma_1 = c_1(Q)$, where Q is the universal quotient bundle of $G(2, 8)$. Then the classical quadruple intersection number is calculated as $\kappa = \int_{X_{1^8}} \sigma_1^4 = 132$, and we also see that $\int_{X_{1^8}} c_3(X_{1^8}) \wedge \sigma_1 = -336$ (for details, see Appendix A).¹⁰

The GLSM which describes $X_{1^8} \subset G(2, 8)$ has $U(2)$ gauge group and matter multiplets given in Table 9 (see [37]). These chiral multiplets are coupled through a superpotential $W = \sum_{a,i,j=1}^8 A_{ij}^a P_a (\Phi_1^j \Phi_2^k - \Phi_2^j \Phi_1^k)$, where A_{ij}^a are eight antisymmetric 8×8 matrices.

Using the formulas (2.14) – (2.17), the partition function of this model is given by

$$Z_{\text{GLSM}} = \frac{1}{2} \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i\theta(m_1+m_2)} \int_{-\infty}^{\infty} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} e^{-4\pi i r(\sigma_1+\sigma_2)} \left[\frac{(m_1 - m_2)^2}{4} + (\sigma_1 - \sigma_2)^2 \right] \\ \times \left[\frac{\Gamma(\mathfrak{q} - i\sigma_1 - \frac{1}{2}m_1)}{\Gamma(1 - \mathfrak{q} + i\sigma_1 - \frac{1}{2}m_1)} \frac{\Gamma(\mathfrak{q} - i\sigma_2 - \frac{1}{2}m_2)}{\Gamma(1 - \mathfrak{q} + i\sigma_2 - \frac{1}{2}m_2)} \frac{\Gamma(1 - 2\mathfrak{q} + i(\sigma_1 + \sigma_2) + \frac{1}{2}(m_1 + m_2))}{\Gamma(2\mathfrak{q} - i(\sigma_1 + \sigma_2) + \frac{1}{2}(m_1 + m_2))} \right]^8. \quad (4.31)$$

Here we consider the Grassmann phase $r \gg 0$ which corresponds to the nonlinear sigma model for $X_{1^8} \subset G(2, 8)$. In this phase, the partition function can be evaluated as

$$Z_{\text{GLSM}} = -\frac{1}{2} (z\bar{z})^{2\mathfrak{q}} \oint \frac{d\epsilon_1}{2\pi i} \frac{d\epsilon_2}{2\pi i} (z\bar{z})^{-\epsilon_1-\epsilon_2} \frac{\pi^8 \sin^8 \pi(\epsilon_1 + \epsilon_2)}{\sin^8(\pi\epsilon_1) \sin^8(\pi\epsilon_2)} \\ \times \left| \sum_{k_1, k_2=0}^{\infty} z^{k_1+k_2} [(k_1 - k_2) - (\epsilon_1 - \epsilon_2)] \frac{\Gamma(1 + (k_1 + k_2) - (\epsilon_1 + \epsilon_2))^8}{\Gamma(1 + k_1 - \epsilon_1)^8 \Gamma(1 + k_2 - \epsilon_2)^8} \right|^2, \quad (4.32)$$

¹⁰The mirror construction of Calabi-Yau complete intersections in $G(k, n)$ was studied in [35] by making use of a flat deformation of Grassmannian $G(k, n)$ to a Gorenstein toric Fano variety [36].

where $z = e^{-2\pi r + i\theta}$, and the complex conjugation does not act on $\epsilon_{1,2}$.

The coefficient of $\log^4 \bar{z}$ is given by

$$\frac{132}{4!}(z\bar{z})^{2q}T^0(z)\overline{T^0(z)}, \quad (4.33)$$

where

$$\begin{aligned} T^0(z) &= \sum_{k_1, k_2=0}^{\infty} z^{k_1+k_2} \frac{((k_1+k_2)!)^8}{(k_1!k_2!)^8} [1 - 4(k_1 - k_2)(\Psi(1+k_1) - \Psi(1+k_2))] \\ &= 1 - 6z + 234z^2 - 13164z^3 + 936810z^4 - 76041756z^5 + \dots, \end{aligned} \quad (4.34)$$

We see that the above series expansion of $T^0(-z)$ agrees with the fundamental period

$$\widehat{T}^0(z) = \sum_{\ell_0, \ell_1, \dots, \ell_5=0}^{\infty} z^{\ell_0} \binom{\ell_0}{\ell_1} \binom{\ell_2}{\ell_1} \binom{\ell_0}{\ell_2} \binom{\ell_3}{\ell_2} \binom{\ell_0}{\ell_3} \binom{\ell_4}{\ell_3} \binom{\ell_0}{\ell_4} \binom{\ell_5}{\ell_4} \binom{\ell_0}{\ell_5}^2 \quad (4.35)$$

of the mirror manifold of $X_{18} \subset G(2, 8)$ which can be obtained by means of a prescription of [35] (see also [38]). In Appendix A, we will revisit this issue and refer to a generalization. As noted in [11], it is interesting to prove this type of coincidence from the viewpoint of combinatorics.

Before proceeding with our computation, let us reconsider the problem from the viewpoint of mirror symmetry. We can check that the fundamental period (4.35) is a kernel of the following Picard-Fuchs operator

$$\begin{aligned} \mathcal{D} &= 121(\Theta - 1)\Theta^5 - 22z(438\Theta^5 + 2094\Theta^4 + 1710\Theta^3 + 950\Theta^2 + 275\Theta + 33)\Theta \\ &\quad - z^2(839313\Theta^6 + 2471661\Theta^5 + 4037556\Theta^4 + 4497304\Theta^3 + 3093948\Theta^2 + 1158740\Theta \\ &\quad + 180048) - 2z^3(5746754\Theta^6 + 26470666\Theta^5 + 51184224\Theta^4 + 50480470\Theta^3 \\ &\quad + 26295335\Theta^2 + 6684843\Theta + 604098) - 4z^4(4081884\Theta^6 + 14894484\Theta^5 \\ &\quad + 18825903\Theta^4 + 7472030\Theta^3 - 3698839\Theta^2 - 4099839\Theta - 993618) + 56z^5(29592\Theta^6 \\ &\quad + 255960\Theta^5 + 806448\Theta^4 + 1272787\Theta^3 + 1088403\Theta^2 + 483431\Theta + 87609) \\ &\quad + 1568z^6(4\Theta + 5)(2\Theta + 3)(4\Theta + 3)(\Theta + 1)^3. \end{aligned} \quad (4.36)$$

Therefore, from the rank of this operator, we see that $\dim H_{\text{prim}}^{2,2}(X_{18}) = 2$. This implies the existence of an element H_2 orthogonal to $H_1 \equiv \sigma_1^2$.

After normalizing the partition function Z_{GLSM} by $(z\bar{z})^q(2\pi i)^4 T^0 \overline{T^0}$, we can determine the flat coordinate by picking up the coefficient of $\log^3 \bar{z}$. The result is given by

$$\begin{aligned} 2\pi i t &= \log z + 2\pi i t_{(0)} - \frac{4}{T^0} \sum_{k_1, k_2=0}^{\infty} \frac{((k_1+k_2)!)^8}{(k_1!)^8(k_2!)^8} z^{k_1+k_2} \left[(k_1 - k_2)\Psi^{(1)}(1+k_1) \right. \\ &\quad \left. - 2[\Psi(1+k_1+k_2) - \Psi(1+k_1)] [1 - 4(k_1 - k_2)(\Psi(1+k_1) - \Psi(1+k_2))] \right], \end{aligned} \quad (4.37)$$

d	n_d	m_d
1	1680	336
2	50904	1680
3	2003568	78176
4	108147648	3964128
5	6684193824	241319232
6	456302632296	16148051976
7	33294956299248	1163962252320
8	2553533188012800	88403710299072

Table 10: Gromov-Witten invariants for $X_{1^8} \subset G(2, 8)$.

where $\Psi^{(1)}(x) = \frac{d}{dx}\Psi(x)$. By taking $t_{(0)} = \frac{1}{2}$, we find that the generating function (3.4) associated with H_1 is given by

$$F_1(t) \equiv G(t) = \frac{132}{2}t^2 + \frac{1}{(2\pi i)^2} \sum_{d=1}^{\infty} n_d \text{Li}_2(q^d), \quad (4.38)$$

where the Gromov-Witten invariants n_d are listed in Table 10.

As mentioned above, we can also find a remaining generating function of Gromov-Witten invariants (3.2) associated with the element H_2 :

$$F_2(t) = \frac{c}{(2\pi i)^2} \sum_{d=1}^{\infty} m_d \text{Li}_2(q^d). \quad (4.39)$$

There is an extra integer-valued parameter c associated with the ambiguity of the second diagonal element of the intersection matrix

$$\eta_{mn} = \text{diag}(132, 77c^2) \quad (4.40)$$

defined by H_1 and H_2 . Up to this ambiguity, we can obtain the Gromov-Witten invariants m_d as shown in Table 10. Combining these results, we confirmed that our conjecture (3.3) holds in this example. Furthermore, we also checked that (3.8) are precisely the kernels of the Picard-Fuchs operator (4.36) up to z^8 .

5 Local toric Calabi-Yau varieties

In this section, we consider d dimensional local toric Calabi-Yau varieties with Kähler parameters r_ℓ . Here $\ell = 1, \dots, n-d$. Each of these varieties has an $\mathcal{N} = (2, 2)$ abelian

GLSM description and can be constructed by the symplectic quotient

$$X = \left\{ (\phi_1, \dots, \phi_n) \in \mathbb{C}^n \left| \sum_{i=1}^n Q_i^\ell |\phi_i|^2 = r_\ell, \sum_{i=1}^n Q_i^\ell = 0 \right. \right\} / U(1)^{n-d}, \quad (5.1)$$

where $Q_i^\ell \in \mathbb{Z}$ are $n - d$ charge vectors with n components. The $U(1)^{n-d}$ gauge group acts on the complex scalar fields ϕ_i as $\phi_i \rightarrow e^{i \sum_\ell \epsilon_\ell Q_i^\ell} \phi_i$.

Let us consider whether the relation (2.12) proposed in [11] can be applied to the local toric Calabi-Yau cases. In fact, by taking the geometric engineering limit [39] for the local Hirzebruch surface $\mathcal{O}(-2, -2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, the Seiberg-Witten Kähler potential for $\mathcal{N} = 2$ pure $SU(2)$ SYM has been obtained from the GLSM calculation in [28] (see also Section 5.1.2). Due to the non-compactness of local toric varieties, as suggested in [28], we need to take all the R-charges of the chiral superfields in the corresponding GLSM to be zero. Although the partition function Z_{GLSM} on S^2 diverges under this “non-compact limit”, the authors of [28] claimed that the correct Seiberg-Witten Kähler potential can be obtained from the regular part of Z_{GLSM} . As a generalization of their result, here we claim that the Kähler potential K on the Kähler moduli space of local toric Calabi-Yau varieties can be obtained by

$$e^{-K} = \oint \frac{d\mathbf{q}_1}{2\pi i \mathbf{q}_1} \cdots \oint \frac{d\mathbf{q}_m}{2\pi i \mathbf{q}_m} Z_{\text{GLSM}}, \quad (5.2)$$

up to the degrees of freedom of Kähler transformation. $\{\mathbf{q}_i\}_{i=1}^m$ are the R-charges of m chiral superfields corresponding to the non-compact directions. In the remaining part of this section, we will check our claim in several examples of threefolds and fourfolds. During the computation, we treat the R-charges of the chiral superfields related to the non-compact directions as regulators of the divergence.

5.1 Threefolds

Here we consider local toric Calabi-Yau threefolds. By performing the exact calculation of GLSM partition functions and using the explicit form of the Kähler potential (2.11), we can study their topological nature and also confirm the consistency of our claim (5.2).

5.1.1 Resolved conifold: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$

The GLSM corresponding to the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ can be characterized by the charge vector $Q = (-1, -1, 1, 1)$. Assigning the same R-charge $2\mathbf{q}$ to two chiral superfields corresponding to two non-compact directions, the GLSM partition

function (2.14) is evaluated as

$$\begin{aligned} Z_{\text{GLSM}} &= \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \frac{\Gamma(-i\sigma - \frac{1}{2}m)^2}{\Gamma(1 + i\sigma - \frac{1}{2}m)^2} \frac{\Gamma(\mathfrak{q} + i\sigma + \frac{1}{2}m)^2}{\Gamma(1 - \mathfrak{q} - i\sigma + \frac{1}{2}m)^2} \\ &= \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{-\epsilon} \frac{\sin^2 \pi(\mathfrak{q} - \epsilon)}{\sin^2(\pi\epsilon)} \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(\mathfrak{q} + k - \epsilon)^2}{\Gamma(1 + k - \epsilon)^2} \right|^2, \end{aligned} \quad (5.3)$$

where $z = e^{-2\pi r + i\theta}$ and the complex conjugation does not act on ϵ .

First we normalize the partition function as

$$\tilde{Z}_{\text{GLSM}} = \frac{g(z)\overline{g(z)}}{f(z)\overline{f(z)}} Z_{\text{GLSM}}. \quad (5.4)$$

Here the normalization factors are

$$f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\mathfrak{q} + k)^2}{\Gamma(1 + k)^2} z^k, \quad (5.5)$$

$$g(z) = \tilde{\Gamma}(\mathfrak{q}, h(z))^2 \left[1 + \mathfrak{q}^2 \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(1 + k)^2} \prod_{j=1}^{k-1} (\mathfrak{q} + j)^2 \right], \quad (5.6)$$

where

$$\tilde{\Gamma}(\mathfrak{q}, h(z)) = \sum_{n=0}^{\infty} \frac{\Gamma^{(n)}(1)}{n!} \bigg|_{\gamma=h(z)} \mathfrak{q}^{n-1}, \quad (5.7)$$

and γ is the Euler constant. Note that $\tilde{\Gamma}(\mathfrak{q}, \gamma) = \Gamma(\mathfrak{q})$, and thus $g(z)|_{h(z)=\gamma} = f(z)$. Under the non-compact limit $\mathfrak{q} \rightarrow 0^+$ [28], this normalization only replaces γ with a holomorphic function $h(z)$. This prescription is necessary to produce the classical term of the Kähler potential. Then the behavior of the partition function under the non-compact limit is given by

$$\begin{aligned} \tilde{Z}_{\text{GLSM}} &= 2\mathfrak{q}^{-3} - (4h(z) + 4\overline{h(z)} + \log z\bar{z})\mathfrak{q}^{-2} \\ &\quad + 2(h(z) + \overline{h(z)})(2h(z) + 2\overline{h(z)} + \log z\bar{z})\mathfrak{q}^{-1} + \tilde{Z}_0 + \mathcal{O}(\mathfrak{q}), \end{aligned} \quad (5.8)$$

where

$$\begin{aligned} \tilde{Z}_0 &= -\frac{8}{3}(h(z) + \overline{h(z)})^3 - 2(h(z) + \overline{h(z)})^2 \log z\bar{z} + \frac{4}{3}\zeta(3) \\ &\quad - (\text{Li}_2(z) + \text{Li}_2(\bar{z})) \log z\bar{z} + 2(\text{Li}_3(z) + \text{Li}_3(\bar{z})). \end{aligned} \quad (5.9)$$

By taking $h(z) = -\frac{1}{4} \log z$, we find that (5.9) gives the Kähler potential of the form (2.11) with the “natural classical triple intersection number” $\kappa = \frac{1}{2}$ [40]. This result is consistent with our statement (5.2). In a similar way to the case of the compact Calabi-Yau in [11], we can also read off the flat coordinate t and the Gromov-Witten invariants n_d as

$$2\pi i t = \log z, \quad n_1 = 1, \quad n_{d \geq 2} = 0. \quad (5.10)$$

5.1.2 Local Hirzebruch surface: $\mathcal{O}(-2, -2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$

Let us consider the local Hirzebruch surface $\mathcal{O}(-2, -2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ defined by two charge vectors $Q^1 = (-2, 1, 1, 0, 0)$ and $Q^2 = (-2, 0, 0, 1, 1)$. The exact partition function of the corresponding GLSM with an R-charge \mathfrak{q} is given by

$$\begin{aligned} Z_{\text{GLSM}} &= \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i(\theta_1 m_1 + \theta_2 m_2)} \int_{-\infty}^{\infty} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} e^{-4\pi i(r_1 \sigma_1 + r_2 \sigma_2)} \\ &\quad \times \frac{\Gamma(-i\sigma_1 - \frac{1}{2}m_1)^2}{\Gamma(1 + i\sigma_1 - \frac{1}{2}m_1)^2} \frac{\Gamma(-i\sigma_2 - \frac{1}{2}m_2)^2}{\Gamma(1 + i\sigma_2 - \frac{1}{2}m_2)^2} \frac{\Gamma(\mathfrak{q} + 2i(\sigma_1 + \sigma_2) + (m_1 + m_2))}{\Gamma(1 - \mathfrak{q} - 2i(\sigma_1 + \sigma_2) + (m_1 + m_2))} \\ &= \oint \frac{d\epsilon_1}{2\pi i} \frac{d\epsilon_2}{2\pi i} (z_1 \bar{z}_1)^{-\epsilon_1} (z_2 \bar{z}_2)^{-\epsilon_2} \frac{\pi^3 \sin \pi(\mathfrak{q} - 2(\epsilon_1 + \epsilon_2))}{\sin^2(\pi\epsilon_1) \sin^2(\pi\epsilon_2)} \\ &\quad \times \left| \sum_{k_1, k_2=0}^{\infty} z_1^{k_1} z_2^{k_2} \frac{\Gamma(\mathfrak{q} + 2(k_1 + k_2) - 2(\epsilon_1 + \epsilon_2))}{\Gamma(1 + k_1 - \epsilon_1)^2 \Gamma(1 + k_2 - \epsilon_2)^2} \right|^2, \end{aligned} \quad (5.11)$$

where $z_\ell = e^{-2\pi r_\ell + i\theta_\ell}$ and the complex conjugation does not act on $\epsilon_{1,2}$. As in the case of the resolved conifold, we take the non-compact limit $\mathfrak{q} \rightarrow 0^+$ of a normalized partition function $\tilde{Z}_{\text{GLSM}} = \frac{g(z_1, z_2) \overline{g(z_1, z_2)}}{f(z_1, z_2) \overline{f(z_1, z_2)}} Z_{\text{GLSM}}$, where

$$f(z_1, z_2) = \sum_{k_1, k_2=0}^{\infty} \frac{\Gamma(\mathfrak{q} + 2(k_1 + k_2))}{\Gamma(1 + k_1)^2 \Gamma(1 + k_2)^2} z_1^{k_1} z_2^{k_2}, \quad (5.12)$$

$$g(z_1, z_2) = \tilde{\Gamma}(\mathfrak{q}, h(z)) \left[1 + \mathfrak{q} \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{z_1^{k_1} z_2^{k_2}}{\Gamma(1 + k_1)^2 \Gamma(1 + k_2)^2} \prod_{j=1}^{2k_1+2k_2-1} (\mathfrak{q} + j) \right], \quad (5.13)$$

and $h(z) \equiv h(z_1, z_2)$ is a holomorphic function of $z_{1,2}$. Then we obtain the Laurent expansion

$$\begin{aligned} \tilde{Z}_{\text{GLSM}} &= 8\mathfrak{q}^{-3} - 2(4h(z) + 4\overline{h(z)} + \log z_1 z_2 + \log \bar{z}_1 \bar{z}_2) \mathfrak{q}^{-2} \\ &\quad + (2h(z) + 2\overline{h(z)} + \log z_1 \bar{z}_1) (2h(z) + 2\overline{h(z)} + \log z_2 \bar{z}_2) \mathfrak{q}^{-1} + \tilde{Z}_0 + \mathcal{O}(\mathfrak{q}), \end{aligned} \quad (5.14)$$

where \tilde{Z}_0 is given by

$$\begin{aligned} \tilde{Z}_0 &= -\frac{4}{3} (h(z) + \overline{h(z)})^3 - (h(z) + \overline{h(z)})^2 \log z_1 \bar{z}_1 z_2 \bar{z}_2 - (h(z) + \overline{h(z)}) \log z_1 \bar{z}_1 \log z_2 \bar{z}_2 \\ &\quad + (\Delta_{00}(z) + \overline{\Delta_{00}(z)}) \log z_1 \bar{z}_1 \log z_2 \bar{z}_2 + 2\Delta_{00}(z) \overline{\Delta_{00}(z)} \log z_1 \bar{z}_1 z_2 \bar{z}_2 - \frac{16}{3} \zeta(3) \\ &\quad + (\Delta_{01}(z) + \overline{\Delta_{01}(z)}) \log z_1 \bar{z}_1 + (\Delta_{10}(z) + \overline{\Delta_{10}(z)}) \log z_2 \bar{z}_2 - \frac{2}{3} \pi^2 (\Delta_{00}(z) + \overline{\Delta_{00}(z)}) \\ &\quad + 2\Delta_{00}(z) (\overline{\Delta_{10}(z)} + \overline{\Delta_{01}(z)}) + 2\overline{\Delta_{00}(z)} (\Delta_{10}(z) + \Delta_{01}(z)) + \Delta_{11}(z) + \overline{\Delta_{11}(z)}. \end{aligned} \quad (5.15)$$

In the above expression, we have defined $\Delta_{00}(z) \equiv \Delta_{00}(z_1, z_2)$, $\Delta_{10}(z) \equiv \Delta_{10}(z_1, z_2)$, $\Delta_{01}(z) \equiv \Delta_{01}(z_1, z_2) = \Delta_{10}(z_2, z_1)$, and $\Delta_{11}(z) \equiv \Delta_{11}(z_1, z_2)$ as

$$\Delta_{00}(z) = \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{(2k_1 + 2k_2 - 1)!}{(k_1!)^2 (k_2!)^2} z_1^{k_1} z_2^{k_2}, \quad (5.16)$$

$$\Delta_{10}(z) = 2 \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{(2k_1 + 2k_2 - 1)!}{(k_1!)^2 (k_2!)^2} z_1^{k_1} z_2^{k_2} [\Psi(2k_1 + 2k_2) - \Psi(1 + k_1)], \quad (5.17)$$

$$\begin{aligned} \Delta_{11}(z) = 4 \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{(2k_1 + 2k_2 - 1)!}{(k_1!)^2 (k_2!)^2} z_1^{k_1} z_2^{k_2} & \left[\Psi^{(1)}(2k_1 + 2k_2) \right. \\ & \left. + [\Psi(2k_1 + 2k_2) - \Psi(1 + k_1)] [\Psi(2k_1 + 2k_2) - \Psi(1 + k_2)] \right], \quad (5.18) \end{aligned}$$

where $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ and $\Psi^{(1)}(x) = \frac{d}{dx} \Psi(x)$. By choosing $h(z) = -\frac{1}{4} \log(z_1 z_2)$ and comparing (5.15) with (2.11), we obtain the classical triple intersection numbers $\kappa_{111} = \kappa_{222} = \frac{1}{4}$ and $\kappa_{112} = \kappa_{122} = -\frac{1}{4}$ computed in [40, 41]. The flat coordinates are given by

$$2\pi i t^1 = \log z_1 - 2\Delta_{00}(z), \quad 2\pi i t^2 = \log z_2 - 2\Delta_{00}(z), \quad (5.19)$$

and we finally obtain

$$\begin{aligned} \frac{i}{(2\pi i)^3} \tilde{Z}_0 = & -\frac{i}{6} \sum_{\ell, m, n=1,2} \kappa_{\ell mn} (t^\ell - \bar{t}^\ell) (t^m - \bar{t}^m) (t^n - \bar{t}^n) + \frac{2}{3\pi^3} \zeta(3) \\ & - \frac{i}{(2\pi i)^2} (\Delta_{00}(z)^2 - \Delta_{01}(z) + \overline{\Delta_{00}(z)}^2 - \overline{\Delta_{01}(z)}) (t^1 - \bar{t}^1) \\ & - \frac{i}{(2\pi i)^2} (\Delta_{00}(z)^2 - \Delta_{10}(z) + \overline{\Delta_{00}(z)}^2 - \overline{\Delta_{10}(z)}) (t^2 - \bar{t}^2) \\ & + \frac{2i}{(2\pi i)^3} \left(\frac{4}{3} \Delta_{00}(z)^3 - \Delta_{00}(z) (\Delta_{10}(z) + \Delta_{01}(z)) + \Delta_{11}(z) - \frac{\pi^2}{3} \Delta_{00}(z) + c.c. \right). \quad (5.20) \end{aligned}$$

From this result, we can extract the Gromov-Witten invariants $n_{d_1, d_2} = n_{d_2, d_1}$ as

$$n_{1,0} = -2, \quad n_{2,0} = 0, \quad n_{1,1} = -4, \quad n_{2,1} = -6, \quad n_{3,1} = -8, \quad n_{2,2} = -32, \quad n_{3,2} = -110, \dots \quad (5.21)$$

These are in agreement with the computation in [42].

5.2 Fourfolds

Next, we study local toric Calabi-Yau fourfolds. Just like the case of threefolds, using (3.3) and (5.2) we compute the Gromov-Witten invariants for three local examples discussed in [6]. This also corresponds to the nontrivial check for our statements.

5.2.1 Local \mathbb{P}^2 : $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \rightarrow \mathbb{P}^2$

Toric charge of the local Calabi-Yau $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \rightarrow \mathbb{P}^2$ is given by $Q = (-1, -2, 1, 1, 1)$, and we denote the Kähler form defined on the base \mathbb{P}^2 by J . Using this data, we can construct GLSM and the partition function is evaluated as

$$\begin{aligned} Z_{\text{GLSM}} &= \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \\ &\quad \times \frac{\Gamma(-i\sigma - \frac{1}{2}m)^3}{\Gamma(1 + i\sigma - \frac{1}{2}m)^3} \frac{\Gamma(\mathbf{q}_1 + i\sigma + \frac{1}{2}m)}{\Gamma(1 - \mathbf{q}_1 - i\sigma + \frac{1}{2}m)} \frac{\Gamma(\mathbf{q}_2 + 2i\sigma + m)}{\Gamma(1 - \mathbf{q}_2 - 2i\sigma + m)} \\ &= \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{-\epsilon} \frac{\pi \sin \pi(\mathbf{q}_1 - \epsilon) \sin \pi(\mathbf{q}_2 - 2\epsilon)}{\sin^3(\pi\epsilon)} \\ &\quad \times \left| \sum_{k=0}^{\infty} (-z)^k \frac{\Gamma(\mathbf{q}_1 + k - \epsilon) \Gamma(\mathbf{q}_2 + 2k - 2\epsilon)}{\Gamma(1 + k - \epsilon)^3} \right|^2. \end{aligned} \quad (5.22)$$

Here $z = e^{-2\pi r + i\theta}$ and the complex conjugation does not act on $\epsilon_{1,2}$. In a similar way to the above threefold examples, we consider asymptotic behavior of a normalized partition function $\tilde{Z}_{\text{GLSM}} = \frac{g(z)\overline{g(z)}}{f(z)\overline{f(z)}} Z_{\text{GLSM}}$ under the non-compact limit $\mathbf{q}_{1,2} \rightarrow 0^+$. Here $f(z)$ and $g(z)$ are defined by

$$f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\mathbf{q}_1 + k) \Gamma(\mathbf{q}_2 + 2k)}{\Gamma(1 + k)^3} (-z)^k, \quad (5.23)$$

$$g(z) = \tilde{\Gamma}(\mathbf{q}_1, h(z)) \tilde{\Gamma}(\mathbf{q}_2, h(z)) \left[1 + \mathbf{q}_1 \mathbf{q}_2 \sum_{k=1}^{\infty} \frac{(-z)^k}{\Gamma(1 + k)^3} \prod_{j_1=1}^{k-1} (\mathbf{q}_1 + j_1) \cdot \prod_{j_2=1}^{2k-1} (\mathbf{q}_2 + j_2) \right]. \quad (5.24)$$

Performing the double series expansion, we obtain

$$\begin{aligned} \tilde{Z}_{\text{GLSM}} &= \frac{(2\mathbf{q}_1 + \mathbf{q}_2)^2}{\mathbf{q}_1^3 \mathbf{q}_2^3} - (h(z) + \overline{h(z)}) \frac{4\mathbf{q}_1^3 + \mathbf{q}_2^3}{\mathbf{q}_1^3 \mathbf{q}_2^3} - (3h(z) + 3\overline{h(z)} + \log z\bar{z}) \frac{2\mathbf{q}_1 + \mathbf{q}_2}{\mathbf{q}_1^2 \mathbf{q}_2^2} \\ &\quad + (h(z) + \overline{h(z)}) (2h(z) + 2\overline{h(z)} + \log z\bar{z}) \mathbf{q}_1^{-2} + \frac{1}{2} (3h(z) + 3\overline{h(z)} + \log z\bar{z})^2 \mathbf{q}_1^{-1} \mathbf{q}_2^{-1} \\ &\quad + (h(z) + \overline{h(z)}) (5h(z) + 5\overline{h(z)} + 2 \log z\bar{z}) \mathbf{q}_2^{-2} + \tilde{Z}_{10} \mathbf{q}_1^{-1} + \tilde{Z}_{01} \mathbf{q}_2^{-1} + \tilde{Z}_{00} + \dots, \end{aligned} \quad (5.25)$$

where

$$\tilde{Z}_{10} - \tilde{Z}_{01} = (h(z) + \overline{h(z)})^3 + \frac{1}{2} (h(z) + \overline{h(z)})^2 \log z\bar{z} - 2\zeta(3), \quad (5.26)$$

$$\tilde{Z}_{10} - 4\tilde{Z}_{01} = \frac{1}{2} (h(z) + \overline{h(z)}) (3h(z) + 3\overline{h(z)} + \log z\bar{z}) (7h(z) + 7\overline{h(z)} + 3 \log z\bar{z}), \quad (5.27)$$

and

$$\begin{aligned} \tilde{Z}_{00} &= \frac{4}{3} (h(z) + \overline{h(z)})^4 + \frac{3}{2} (h(z) + \overline{h(z)})^3 \log z\bar{z} + (h(z) + \overline{h(z)})^2 (\log z\bar{z})^2 \\ &\quad + \frac{10}{3} \zeta(3) (h(z) + \overline{h(z)}) + \frac{1}{2} (\Delta_0(z) + \overline{\Delta_0(z)}) (\log z\bar{z})^2 - (\Delta_1(z) + \overline{\Delta_1(z)}) \log z\bar{z} \\ &\quad + 2\Delta_0(z) \overline{\Delta_0(z)} + \Delta_2(z) + \overline{\Delta_2(z)} - \frac{1}{6} \pi^2 (\Delta_0(z) + \overline{\Delta_0(z)}). \end{aligned} \quad (5.28)$$

In the above expression, we have defined

$$\Delta_0(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k^2} \binom{2k}{k} z^k, \quad (5.29)$$

$$\Delta_1(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k^3} \binom{2k}{k} z^k - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{2k^2} \binom{2k}{k} z^k [\Psi(2k) - \Psi(1+k)], \quad (5.30)$$

$$\begin{aligned} \Delta_2(z) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k^2} \binom{2k}{k} z^k & \left[[\Psi(k) + 2\Psi(2k) - 3\Psi(1+k)]^2 \right. \\ & \left. + \Psi^{(1)}(k) + 4\Psi^{(1)}(2k) - 3\Psi^{(1)}(1+k) \right]. \end{aligned} \quad (5.31)$$

Let us choose $h(z) = -\frac{1}{2} \log z$. By comparing (5.28) with our conjecture (3.3) for the Kähler potential, we can obtain the “classical quadruple intersection number” $\kappa = \frac{1}{2}$. Furthermore, the flat coordinate t and the generating function (3.4) of the Gromov-Witten invariants associated with J^2 can be also extracted as

$$2\pi it = \log z, \quad G(t) = \frac{1}{4} t^2 + \Delta_0(q), \quad q = e^{2\pi it}. \quad (5.32)$$

This result coincides with the computation in [6]. With the above choice for $h(z)$, the consistent inverse intersection matrix $\eta^{-1} = 2$ for a basis J^2 can be realized. This result provides a nontrivial verification of our conjecture (3.3).

5.2.2 Local $\mathbb{P}^1 \times \mathbb{P}^1$: $\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$

The local Calabi-Yau fourfold $\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is defined by two charge vectors $Q^1 = (-1, -1, 1, 1, 0, 0)$ and $Q^2 = (-1, -1, 0, 0, 1, 1)$. We denote the Kähler forms defined on the base $\mathbb{P}^1 \times \mathbb{P}^1$ by J_1 and J_2 . In this case, the GLSM partition function (2.14) becomes

$$\begin{aligned} Z_{\text{GLSM}} &= \sum_{m_1, m_2 \in \mathbb{Z}} e^{-i(\theta_1 m_1 + \theta_2 m_2)} \int_{-\infty}^{\infty} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} e^{-4\pi i(r_1 \sigma_1 + r_2 \sigma_2)} \\ &\quad \times \frac{\Gamma(-i\sigma_1 - \frac{1}{2}m_1)^2}{\Gamma(1 + i\sigma_1 - \frac{1}{2}m_1)^2} \frac{\Gamma(-i\sigma_2 - \frac{1}{2}m_2)^2}{\Gamma(1 + i\sigma_2 - \frac{1}{2}m_2)^2} \frac{\Gamma(\mathbf{q} + i(\sigma_1 + \sigma_2) + \frac{1}{2}(m_1 + m_2))}{\Gamma(1 - \mathbf{q} - i(\sigma_1 + \sigma_2) + \frac{1}{2}(m_1 + m_2))} \\ &= \oint \frac{d\epsilon_1}{2\pi i} \frac{d\epsilon_2}{2\pi i} (z_1 \bar{z}_1)^{-\epsilon_1} (z_2 \bar{z}_2)^{-\epsilon_2} \frac{\pi^2 \sin^2 \pi(\mathbf{q} - (\epsilon_1 + \epsilon_2))}{\sin^2(\pi\epsilon_1) \sin^2(\pi\epsilon_2)} \\ &\quad \times \left| \sum_{k_1, k_2=0}^{\infty} z_1^{k_1} z_2^{k_2} \frac{\Gamma(\mathbf{q} + (k_1 + k_2) - (\epsilon_1 + \epsilon_2))^2}{\Gamma(1 + k_1 - \epsilon_1)^2 \Gamma(1 + k_2 - \epsilon_2)^2} \right|^2, \end{aligned} \quad (5.33)$$

where $z_\ell = e^{-2\pi r_\ell + i\theta_\ell}$, and the complex conjugation does not act on $\epsilon_{1,2}$. First we normalize the partition function as $\tilde{Z}_{\text{GLSM}} = \frac{g(z_1, z_2)g(\bar{z}_1, \bar{z}_2)}{f(z_1, z_2)f(\bar{z}_1, \bar{z}_2)} Z_{\text{GLSM}}$, where

$$f(z_1, z_2) = \sum_{k_1, k_2=0}^{\infty} \frac{\Gamma(\mathbf{q} + (k_1 + k_2))^2}{\Gamma(1 + k_1)^2 \Gamma(1 + k_2)^2} z_1^{k_1} z_2^{k_2}, \quad (5.34)$$

$$g(z_1, z_2) = \tilde{\Gamma}(\mathbf{q}, h(z))^2 \left[1 + \mathbf{q}^2 \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{z_1^{k_1} z_2^{k_2}}{\Gamma(1 + k_1)^2 \Gamma(1 + k_2)^2} \prod_{j=1}^{k_1+k_2-1} (\mathbf{q} + j)^2 \right]. \quad (5.35)$$

$h(z) \equiv h(z_1, z_2)$ is a holomorphic function of $z_{1,2}$. Taking the non-compact limit $\mathbf{q} \rightarrow 0^+$, the partition function is expanded as

$$\begin{aligned} \tilde{Z}_{\text{GLSM}} = & 6\mathbf{q}^{-4} - 2(6h(z) + 6\overline{h(z)} + \log z_1 z_2 \bar{z}_1 \bar{z}_2) \mathbf{q}^{-3} \\ & + [12(h(z) + \overline{h(z)})^2 + 4(h(z) + \overline{h(z)}) \log z_1 z_2 \bar{z}_1 \bar{z}_2 + \log z_1 \bar{z}_1 \log z_2 \bar{z}_2] \mathbf{q}^{-2} \\ & - 2(h(z) + \overline{h(z)}) (2h(z) + 2\overline{h(z)} + \log z_1 \bar{z}_1) (2h(z) + 2\overline{h(z)} + \log z_2 \bar{z}_2) \mathbf{q}^{-1} \\ & + \tilde{Z}_0 + \mathcal{O}(\mathbf{q}), \end{aligned} \quad (5.36)$$

where

$$\begin{aligned} \tilde{Z}_0 = & 4(h(z) + \overline{h(z)})^4 + \frac{8}{3}(h(z) + \overline{h(z)})^3 \log z_1 \bar{z}_1 z_2 \bar{z}_2 + 2(h(z) + \overline{h(z)})^2 \log z_1 \bar{z}_1 \log z_2 \bar{z}_2 \\ & + (\Delta_{00}(z) + \overline{\Delta_{00}(z)}) \log z_1 \bar{z}_1 \log z_2 \bar{z}_2 - \frac{4}{3} \zeta(3) \log z_1 \bar{z}_1 z_2 \bar{z}_2 \\ & + (\Delta_{01}(z) + \overline{\Delta_{01}(z)}) \log z_1 \bar{z}_1 + (\Delta_{10}(z) + \overline{\Delta_{10}(z)}) \log z_2 \bar{z}_2 \\ & + 2\Delta_{00}(z) \overline{\Delta_{00}(z)} + \Delta_{11}(z) + \overline{\Delta_{11}(z)} - \frac{1}{3} \pi^2 (\Delta_{00}(z) + \overline{\Delta_{00}(z)}). \end{aligned} \quad (5.37)$$

In the above, we have defined $\Delta_{00}(z) \equiv \Delta_{00}(z_1, z_2)$, $\Delta_{10}(z) \equiv \Delta_{10}(z_1, z_2)$, $\Delta_{01}(z) \equiv \Delta_{01}(z_1, z_2) = \Delta_{10}(z_2, z_1)$, and $\Delta_{11}(z) \equiv \Delta_{11}(z_1, z_2)$ as

$$\Delta_{00}(z) = \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{1}{(k_1 + k_2)^2} \binom{k_1 + k_2}{k_1}^2 z_1^{k_1} z_2^{k_2}, \quad (5.38)$$

$$\Delta_{10}(z) = 2 \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{1}{(k_1 + k_2)^2} \binom{k_1 + k_2}{k_1}^2 z_1^{k_1} z_2^{k_2} [\Psi(k_1 + k_2) - \Psi(1 + k_1)], \quad (5.39)$$

$$\begin{aligned} \Delta_{11}(z) = & 2 \sum_{\substack{k_1, k_2=0 \\ (k_1, k_2) \neq (0,0)}}^{\infty} \frac{1}{(k_1 + k_2)^2} \binom{k_1 + k_2}{k_1}^2 z_1^{k_1} z_2^{k_2} [\Psi^{(1)}(k_1 + k_2) \\ & + 2[\Psi(k_1 + k_2) - \Psi(1 + k_1)][\Psi(k_1 + k_2) - \Psi(1 + k_2)]]]. \end{aligned} \quad (5.40)$$

From the comparison of (5.37) with (3.3), we find that the flat coordinates $t_{1,2}$, and the generating function (3.4) of the Gromov-Witten invariants take the form

$$2\pi it_1 = \log z_1, \quad 2\pi it_2 = \log z_2, \quad (5.41)$$

$$\widehat{G}_{11}(t) = \widehat{G}_{22}(t) = 0, \quad \widehat{G}_{12}(t) = \Delta_{00}(q_1, q_2), \quad q_\ell = e^{2\pi i t_\ell}. \quad (5.42)$$

This result agrees with the computations of [6].

Here let us take a basis $\{H_1, H_2, H_3\} = \{J_1^2, J_1 \wedge J_2, J_2^2\}$, and consider the associated generating functions (3.2). Then, we can read off $\eta^{22} = 2$ from (3.3). In order to obtain the “classical quadruple intersection numbers” consistent with this assignment, we need to choose $h(z) = -\frac{1}{4} \log(z_1 z_2)$. With this choice, we obtain $\kappa_{1111} = \kappa_{2222} = -\frac{5}{8}$, $\kappa_{1112} = \kappa_{1222} = \frac{1}{8}$, and $\kappa_{1122} = \frac{3}{8}$.

5.2.3 Local \mathbb{P}^3 : $\mathcal{O}(-4) \rightarrow \mathbb{P}^3$

Finally we consider the local Calabi-Yau fourfold $\mathcal{O}(-4) \rightarrow \mathbb{P}^3$ defined by the charge vector $Q = (-4, 1, 1, 1, 1)$. Let J be the Kähler form defined on the base \mathbb{P}^3 . The two sphere partition function of the corresponding GLSM is given by

$$\begin{aligned} Z_{\text{GLSM}} &= \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} \frac{\Gamma(-i\sigma - \frac{1}{2}m)^4}{\Gamma(1 + i\sigma - \frac{1}{2}m)^4} \frac{\Gamma(\mathbf{q} + 4i\sigma + 2m)}{\Gamma(1 - \mathbf{q} - 4i\sigma + 2m)} \\ &= \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{-\epsilon} \frac{\pi^3 \sin \pi(\mathbf{q} - 4\epsilon)}{\sin^4(\pi\epsilon)} \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(\mathbf{q} + 4k - 4\epsilon)}{\Gamma(1 + k - \epsilon)^4} \right|^2, \end{aligned} \quad (5.43)$$

where $z = e^{-2\pi r + i\theta}$ and the complex conjugation does not act on ϵ . Then, we normalize the partition function as $\widetilde{Z}_{\text{GLSM}} = \frac{g(z)\overline{g(z)}}{f(z)\overline{f(z)}} Z_{\text{GLSM}}$, where

$$f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\mathbf{q} + 4k)}{\Gamma(1 + k)^4} z^k, \quad (5.44)$$

$$g(z) = \widetilde{\Gamma}(\mathbf{q}, h(z)) \left[1 + \mathbf{q} \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(1 + k)^4} \prod_{j=1}^{4k-1} (\mathbf{q} + j) \right]. \quad (5.45)$$

By taking the non-compact limit $\mathbf{q} \rightarrow 0^+$, we obtain the following expansion

$$\begin{aligned} \widetilde{Z}_{\text{GLSM}} &= 64\mathbf{q}^{-4} - 16(4h(z) + 4\overline{h(z)} + \log z\bar{z})\mathbf{q}^{-3} + 2(4h(z) + 4\overline{h(z)} + \log z\bar{z})^2\mathbf{q}^{-2} \\ &\quad - \left[\frac{1}{6}(4h(z) + 4\overline{h(z)} + \log z\bar{z})^3 + \frac{8}{3}\zeta(3) \right] \mathbf{q}^{-1} + \widetilde{Z}_0 + \mathcal{O}(\mathbf{q}). \end{aligned} \quad (5.46)$$

Here we choose $h(z) = -\frac{1}{4} \log z$. By comparing \widetilde{Z}_0 with (3.3), we obtain the “consistent quadruple intersection number” $\kappa = -\frac{1}{4}$ along with the inverse intersection matrix $\eta^{-1} = -4$ for a basis J^2 . We can also find that the topological invariant defined in (3.7) is given

by $C = 5$, and the flat coordinate t and the Gromov-Witten invariants n_d associated with J^2 are determined as

$$2\pi it = \log z + 4 \sum_{k=1}^{\infty} \frac{(4k-1)!}{(k!)^4} z^k = \log z + 24z + 1260z^2 + 123200z^3 + \cdots, \quad (5.47)$$

$$n_d = -20, \quad -820, \quad -68060, \quad -7486440, \quad -965038900, \dots, \quad (5.48)$$

which completely agree with the result of [6].

6 Conclusion and discussions

In this paper we have studied quantum nature of the Kähler moduli space of Calabi-Yau fourfolds. We utilized the recently proposed method which relates the exact two sphere partition function of an $\mathcal{N} = (2, 2)$ GLSM to the Kähler potential on the quantum Kähler moduli space of a Calabi-Yau manifold. Especially we conjectured the explicit formula of the quantum-corrected Kähler potential for Calabi-Yau fourfolds. We also checked our conjecture by computing the genus zero Gromov-Witten invariants and comparing the results with mirror symmetry predictions. Since the GLSM calculation for the Kähler potential is reminiscent of the well-studied abelian mirror symmetry and is also applicable to non-abelian GLSMs, this method would give a clue to understand the non-abelian mirror symmetry.

Moreover, we proposed the local toric analogue of the correspondence between the GLSM partition function and the exact Kähler potential by extending the argument of [28]. We also studied the exact GLSM partition functions for local toric Calabi-Yau varieties in a similar manner to the cases of compact Calabi-Yau manifold. In order to realize the expected classical terms (intersection numbers) of the Kähler potential for a local toric Calabi-Yau variety, we need to modify the normalization of the corresponding GLSM partition function. In this normalization, we have introduced a holomorphic function $h(z)$ which should take the form $h(z) = \sum_{\ell} c_{\ell} \log z_{\ell}$. For local toric Calabi-Yau fourfolds, we fixed the constants c_{ℓ} from the consistency with the intersection matrix on $H_{\text{prim}}^{2,2}$ appeared in our conjectural formula for the Kähler potential.

An immediate generalization of our work is to find the exact Kähler potential for higher dimensional Calabi-Yau manifolds with $d \geq 5$. Once the explicit formula is found, the GLSM calculation would provide an efficient way to compute the Gromov-Witten invariants of Calabi-Yau manifold with arbitrary dimension.

As demonstrated in [43], the exact GLSM partition function is also useful to study the Landau-Ginzburg phase of GLSM which describes a Calabi-Yau manifold at the large

radius point. It would be also interesting to study such a phase transition using the GLSM partition function for not only three dimension but also higher dimensions.

Our conjecture about the exact Kähler potential for Calabi-Yau fourfold allows one to study the nonperturbative aspects of the F-theory compactification. In contrast to the Type IIB string compactification, corrections to the tree level Kähler potential has yet to be fully understood in the F-theory compactification. We hope that many applications of our result would reveal themselves.

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A Note on Grassmannian Calabi-Yau manifold

In this appendix, we first review the computation of the classical cohomology ring of the Grassmannian $G(k, n)$. Then we consider the GLSM description of general complete intersection Calabi-Yau manifold in $G(k, n)$ and generalize the relation between (4.34) and (4.35). We also summarize some computational results for topological invariants of Grassmannian Calabi-Yau fourfold.

A.1 Schubert calculus and Chern classes of Grassmannian

The Grassmannian $G(k, n)$ is defined by the set of k -planes Λ in \mathbb{C}^n . Bases of Λ consist of kn components up to the $GL(k, \mathbb{C})$ action on Λ . Thus the dimension of $G(k, n)$ is given by $k(n - k)$. The cohomology ring of $G(k, n)$ is described by the classes of the Schubert cycles

$$\sigma_{a_1, \dots, a_k}(V) = \{ \Lambda \in G(k, n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i, \ i = 1, \dots, k \}, \quad (\text{A.1})$$

which generate the integral homology (see, for example, [44]). Here $V = (V_1 \subset V_2 \subset \dots \subset V_n)$ is a flag composed by i -dimensional subspaces V_i in \mathbb{C}^n and the index a_i is an integer sequence satisfying $n - k \geq a_1 \geq a_2 \geq \dots \geq a_k \geq 0$. The codimension of Schubert cycle $\sigma_{\vec{a}} \equiv \sigma_{a_1, \dots, a_k}(V)$ is given by $|\vec{a}| \equiv \sum_{i=1}^k a_i$.

The intersection number of Schubert cycles can be computed by Pieri's formula

$$\sigma_a \cdot \sigma_{\vec{b}} = \sum_{\substack{b_i \leq c_i \leq b_{i-1} \\ |\vec{c}| = a + |\vec{b}|}} \sigma_{\vec{c}}. \quad (\text{A.2})$$

Here $\sigma_a \equiv \sigma_{a,0,\dots,0}(V)$ is called the special Schubert cycle, and in the following we summarize some results of their intersection numbers. Note that, in order to define the intersection number of Schubert cycles $\sigma_{\vec{a}_\ell}$, $\ell = 1, \dots, p$, we need to require that the codimension of the union of those cycles is equal to the dimension of $G(k, n)$, i.e. $\sum_{\ell=1}^p |\vec{a}_\ell| = k(n-k)$.

$G(2, 5)$:

$$\sigma_1^6 = \sigma_1^4(\sigma_2 + \sigma_{1,1}) = \sigma_1^3(\sigma_3 + 2\sigma_{2,1}) = \sigma_1^2(3\sigma_{3,1} + 2\sigma_{2,2}) = \sigma_1 \cdot 5\sigma_{3,2} = 5\sigma_{3,3} = 5. \quad (\text{A.3})$$

Similarly, we can obtain

$$\sigma_1^4\sigma_2 = 3, \quad \sigma_1^3\sigma_3 = 1, \quad \sigma_1^2\sigma_2^2 = 2. \quad (\text{A.4})$$

$G(2, 6)$:

$$\sigma_1^8 = 14, \quad \sigma_1^6\sigma_2 = 9, \quad \sigma_1^5\sigma_3 = 4, \quad \sigma_1^4\sigma_2^2 = 6, \quad \sigma_1^4\sigma_4 = 1. \quad (\text{A.5})$$

$G(2, 7)$:

$$\sigma_1^{10} = 42, \quad \sigma_1^8\sigma_2 = 28, \quad \sigma_1^7\sigma_3 = 14, \quad \sigma_1^6\sigma_2^2 = 19, \quad \sigma_1^6\sigma_4 = 5. \quad (\text{A.6})$$

$G(2, 8)$:

$$\sigma_1^{12} = 132, \quad \sigma_1^{10}\sigma_2 = 90, \quad \sigma_1^9\sigma_3 = 48, \quad \sigma_1^8\sigma_4 = 20, \quad \sigma_1^8\sigma_2^2 = 62. \quad (\text{A.7})$$

$G(3, 6)$:

$$\sigma_1^9 = 42, \quad \sigma_1^7\sigma_2 = 21, \quad \sigma_1^6\sigma_3 = 5, \quad \sigma_1^5\sigma_2^2 = 11. \quad (\text{A.8})$$

In the following, we denote the Poincaré dual of the special Schubert cycle by the same symbol σ_a . The total Chern class of Grassmannian $G(k, n)$ is given by [45, 46]

$$c(G(k, n)) = \prod_{i=1}^{n-k} (1 - x_i)^n \cdot \prod_{i,j=1}^{n-k} (1 - (x_i - x_j)^2)^{-\frac{1}{2}}, \quad (\text{A.9})$$

where the p -th Chern class $c_p(G(k, n))$ is obtained as a coefficient of h^p after changing all the variables x_i into hx_i and taking the series expansion of h . Then $c_p(G(k, n))$ can be expressed in terms of the elementary symmetric polynomials $e_a(x_i) = \sum_{i_1 < \dots < i_a} x_{i_1} \cdots x_{i_a}$ which are identified with the cohomology classes σ_a . In the following, we summarize the results for several examples.

$G(2, 5)$:

$$\begin{aligned} c_1(G(2, 5)) &= 5\sigma_1, \quad c_2(G(2, 5)) = 12\sigma_1^2 - \sigma_2, \quad c_3(G(2, 5)) = 20\sigma_1^3 - 10\sigma_1\sigma_2 + 5\sigma_3, \\ c_4(G(2, 5)) &= 28\sigma_1^4 - 38\sigma_1^2\sigma_2 + 20\sigma_1\sigma_3 + 7\sigma_2^2 - 210\sigma_4. \end{aligned} \quad (\text{A.10})$$

Field	$U(k)$	$U(1)_V$
Φ^i	\mathbf{k}_{+1}	$2\mathbf{q}$
P_a	$\mathbf{1}_{-kd_a}$	$2 - 2kd_a\mathbf{q}$

Table 11: Matter content of the $U(k)$ GLSM describing the Grassmannian Calabi-Yau d -fold $X_{d_1, \dots, d_r} \subset G(k, n)$. Here $i = 1, \dots, n$ and $a = 1, \dots, r$. The subscript denotes the charge under the central $U(1) \subset U(k)$.

$G(2, 6)$:

$$\begin{aligned} c_1(G(2, 6)) &= 6\sigma_1, \quad c_2(G(2, 6)) = 18\sigma_1^2 - 2\sigma_2, \quad c_3(G(2, 6)) = 38\sigma_1^3 - 18\sigma_1\sigma_2 + 6\sigma_3, \\ c_4(G(2, 6)) &= 66\sigma_1^4 - 74\sigma_1^2\sigma_2 + 32\sigma_1\sigma_3 + 9\sigma_2^2 - 2\sigma_4. \end{aligned} \quad (\text{A.11})$$

$G(2, 7)$:

$$\begin{aligned} c_1(G(2, 7)) &= 7\sigma_1, \quad c_2(G(2, 7)) = 25\sigma_1^2 - 3\sigma_2, \quad c_3(G(2, 7)) = 63\sigma_1^3 - 28\sigma_1\sigma_2 + 7\sigma_3, \\ c_4(G(2, 7)) &= 129\sigma_1^4 - 127\sigma_1^2\sigma_2 + 46\sigma_1\sigma_3 + 12\sigma_2^2 - 3\sigma_4. \end{aligned} \quad (\text{A.12})$$

$G(2, 8)$:

$$\begin{aligned} c_1(G(2, 8)) &= 8\sigma_1, \quad c_2(G(2, 8)) = 33\sigma_1^2 - 4\sigma_2, \quad c_3(G(2, 8)) = 96\sigma_1^3 - 40\sigma_1\sigma_2 + 8\sigma_3, \\ c_4(G(2, 8)) &= 225\sigma_1^4 - 200\sigma_1^2\sigma_2 + 62\sigma_1\sigma_3 + 16\sigma_2^2 - 4\sigma_4. \end{aligned} \quad (\text{A.13})$$

$G(3, 6)$:

$$\begin{aligned} c_1(G(3, 6)) &= 6\sigma_1, \quad c_2(G(3, 6)) = 17\sigma_1^2, \quad c_3(G(3, 6)) = 32\sigma_1^3 - 6\sigma_1\sigma_2 + 6\sigma_3, \\ c_4(G(3, 6)) &= 48\sigma_1^4 - 36\sigma_1^2\sigma_2 + 30\sigma_1\sigma_3 + 6\sigma_2^2 - 372\sigma_4. \end{aligned} \quad (\text{A.14})$$

A.2 GLSM description for Grassmannian Calabi-Yau manifold

Let us consider a d dimensional Calabi-Yau manifold X_{d_1, \dots, d_r} defined by a complete intersection of r hyperplanes with degrees (d_1, \dots, d_r) in the Grassmannian $G(k, n)$. The complex dimension of X_{d_1, \dots, d_r} is given by $d = kn - k^2 - r$ and the Calabi-Yau condition $d_1 + \dots + d_r = n$ must be satisfied. The total Chern class of this manifold is [46]

$$c(X_{d_1, \dots, d_r}) = \frac{c(G(k, n))}{(1 + d_1\sigma_1) \cdots (1 + d_r\sigma_1)}. \quad (\text{A.15})$$

Using this formula and (A.9), one can compute the classical topological invariants. For example, the intersection number and the Euler characteristic are given by

$$\kappa = \int_{X_{d_1, \dots, d_r}} \sigma_1^d = \int_{G(k, n)} \sigma_1^d \wedge \prod_{a=1}^r d_a \sigma_1, \quad (\text{A.16})$$

$$\chi = \int_{X_{d_1, \dots, d_r}} c_d(X_{d_1, \dots, d_r}) = \int_{G(k, n)} c_d(X_{d_1, \dots, d_r}) \wedge \prod_{a=1}^r d_a \sigma_1. \quad (\text{A.17})$$

The Grassmannian Calabi-Yau d -fold $X_{d_1, \dots, d_r} \subset G(k, n)$ can be described by the $U(k)$ GLSM with matter multiplets shown in Table 11. The superpotential is given by $W = \sum_{a=1}^r P_a W_{d_a}(B)$, where $W_{d_a}(B)$ is a degree d_a polynomial in the baryonic variables¹¹ $B_{i_1 \dots i_k} = \epsilon_{I_1 \dots I_k} \Phi_{i_1}^{I_1} \dots \Phi_{i_k}^{I_k}$ [37]. Then we can compute the two sphere partition function (2.14) in the same way as in Section 4.4. In the Grassmann phase $r \gg 0$, we obtain

$$\begin{aligned} Z_{\text{GLSM}} &= (-1)^{\frac{1}{2}k(k-1)} \frac{1}{k!} (z\bar{z})^{kq} \oint \frac{d\epsilon_1 \dots d\epsilon_k}{(2\pi i)^k} (z\bar{z})^{-\sum_{i=1}^k \epsilon_i} \frac{\pi^{kn-r} \prod_{a=1}^r \sin(\pi d_a \sum_{i=1}^k \epsilon_i)}{\prod_{i=1}^k \sin^n(\pi \epsilon_i)} \\ &\times \left| \sum_{\ell_1, \dots, \ell_k=0}^{\infty} ((-1)^n z)^{\sum_{i=1}^k \ell_i} \prod_{1 \leq i < j \leq k} [(\ell_i - \ell_j) - (\epsilon_i - \epsilon_j)] \cdot \frac{\prod_{a=1}^r \Gamma(1 + d_a \sum_{i=1}^k (\ell_i - \epsilon_i))}{\prod_{i=1}^k \Gamma(1 + \ell_i - \epsilon_i)^n} \right|^2, \end{aligned} \quad (\text{A.18})$$

where $z = e^{-2\pi r + i\theta}$.

Here we consider the case of $k = 2$. As demonstrated in Section 4, we can extract a holomorphic function

$$\begin{aligned} T^0(z) &= \sum_{k_1, k_2=0}^{\infty} (-z)^{k_1+k_2} \frac{(d_1(k_1+k_2))! \dots (d_r(k_1+k_2))!}{(k_1! k_2!)^n} \\ &\times \left[1 - \frac{n}{2} (k_1 - k_2) (\Psi(1+k_1) - \Psi(1+k_2)) \right], \end{aligned} \quad (\text{A.19})$$

which gives a normalization of the partition function (A.18). On the other hand, the fundamental period of the corresponding mirror manifold is given by [35, 38]

$$\hat{T}^0(z) = \sum_{\ell_0, \ell_1, \dots, \ell_{n-3}=0}^{\infty} z^{\ell_0} \frac{(d_1 \ell_0)! \dots (d_r \ell_0)!}{(\ell_0!)^n} \prod_{i=1}^{n-3} \binom{\ell_0}{\ell_i} \binom{\ell_{i+1}}{\ell_i}, \quad (\text{A.20})$$

where we defined $\ell_{n-2} \equiv \ell_0$. We can check the coincidence between $T^0(z)$ and $\hat{T}^0(z)$ up to higher order in z and thus the identity

$$T^0(z) = \hat{T}^0(z) \quad (\text{A.21})$$

¹¹These variables are called Plücker coordinates corresponding to the homogeneous coordinates on the projective space in which the Grassmannian is embedded.

is expected to hold exactly for $d_a \geq 0$ and $n \geq 3$. Note that the Calabi-Yau condition $d_1 + \dots + d_r = n$ need not be satisfied for this identity. Generalizing this identity to arbitrary k is straightforward. It will be interesting to prove this identity from the perspective of combinatorics.

A.3 Gromov-Witten invariants of Grassmannian fourfolds

In $d = 4$, we can list all the Grassmannian Calabi-Yau manifold $X_{d_1, \dots, d_r} \subset G(k, n)$ as

$$\begin{aligned} X_{1,4} &\subset G(2, 5), \quad X_{2,3} \subset G(2, 5), \quad X_{1^3,3} \subset G(2, 6), \\ X_{1^2,2^2} &\subset G(2, 6), \quad X_{1^5,2} \subset G(2, 7), \quad X_{1^4,2} \subset G(3, 6), \end{aligned} \quad (\text{A.22})$$

including $X_{1^8} \subset G(2, 8)$ discussed in Section 4.4. We checked that our conjecture (3.3) holds in these cases, and the Gromov-Witten invariants associated with σ_1^2 are summarised in Table 12.

In a similar way to the example X_{1^8} in Section 4.4, for each example except for $X_{1^4,2} \subset G(3, 6)$ we find a remaining generating function (3.2) defined for an element H_2 orthogonal to $H_1 = \sigma_1^2$. As we mentioned around (4.39), aside from the ambiguity $c \in \mathbb{Z}$ in the intersection matrix, we can obtain the corresponding Gromov-Witten invariants. The results are summarized in Table 13. The classical topological invariants¹², the Picard-Fuchs operator, and the intersection matrix are listed below.

$X_{1,4} \subset G(2, 5)$:

$$\begin{aligned} \chi &= 1848, \quad \int_{X_{1,4}} c_3 \wedge \sigma_1 = -440, \quad \int_{X_{1,4}} c_2 \wedge \sigma_1^2 = 148, \quad \kappa = 20, \\ h^{1,1} &= 1, \quad h^{2,1} = 0, \quad h^{2,2} = 1244, \quad h^{3,1} = 299, \quad \eta_{mn} = \text{diag}(20, 5c^2), \end{aligned} \quad (\text{A.23})$$

$$\begin{aligned} \mathcal{D} &= (\Theta - 1)\Theta^5 - 8z(4\Theta + 3)(4\Theta + 1)(2\Theta + 1)(11\Theta^2 + 11\Theta + 3)\Theta \\ &\quad - 64z^2(4\Theta + 7)(4\Theta + 5)(4\Theta + 3)(4\Theta + 1)(2\Theta + 3)(2\Theta + 1). \end{aligned} \quad (\text{A.24})$$

$X_{2,3} \subset G(2, 5)$:

$$\begin{aligned} \chi &= 1188, \quad \int_{X_{2,3}} c_3 \wedge \sigma_1 = -360, \quad \int_{X_{2,3}} c_2 \wedge \sigma_1^2 = 162, \quad \kappa = 30, \\ h^{1,1} &= 1, \quad h^{2,1} = 0, \quad h^{2,2} = 804, \quad h^{3,1} = 189, \quad \eta_{mn} = \text{diag}(30, 30c^2), \end{aligned} \quad (\text{A.25})$$

$$\begin{aligned} \mathcal{D} &= (\Theta - 1)\Theta^5 - 6z(3\Theta + 2)(3\Theta + 1)(2\Theta + 1)(11\Theta^2 + 11\Theta + 3)\Theta \\ &\quad - 36z^2(3\Theta + 5)(3\Theta + 4)(3\Theta + 2)(3\Theta + 1)(2\Theta + 3)(2\Theta + 1). \end{aligned} \quad (\text{A.26})$$

¹²Method used in calculating the Hodge numbers is explained e.g. in [5].

	$X_{1,4} \subset G(2, 5)$	$X_{2,3} \subset G(2, 5)$	$X_{1^3,3} \subset G(2, 6)$
n_1	9440	5580	4158
n_2	4383680	1102770	538272
n_3	3701308960	391989240	115394706
n_4	4126541676160	183418036920	32820139926
n_5	5368332901844000	100068916666500	10856106949968
	$X_{1^2,2^2} \subset G(2, 6)$	$X_{1^5,2} \subset G(2, 7)$	$X_{1^4,2} \subset G(3, 6)$
n_1	3136	2296	2520
n_2	242032	112196	112140
n_3	30787008	8076880	8494920
n_4	5179177248	781233880	829679760
n_5	1012577938176	87311729064	94209368400

Table 12: Gromov-Witten invariants for Grassmannian Calabi-Yau fourfolds.

	$X_{1,4} \subset G(2, 5)$	$X_{2,3} \subset G(2, 5)$	$X_{1^3,3} \subset G(2, 6)$	$X_{1^2,2^2} \subset G(2, 6)$	$X_{1^5,2} \subset G(2, 7)$
m_1	240	360	252	336	112
m_2	36480	21240	7686	5712	980
m_3	31996560	7998480	1929564	865872	89152
m_4	34141832160	3580395840	518431788	137583600	8067556
m_5	43380207546000	1907243811000	167753878488	26310378528	884735376

Table 13: Remaining Gromov-Witten invariants for Grassmannian Calabi-Yau fourfolds up to the ambiguity $c \in \mathbb{Z}$ in (4.39).

$X_{1^3,3} \subset G(2, 6)$:

$$\chi = 1368, \quad \int_{X_{1^3,3}} c_3 \wedge \sigma_1 = -426, \quad \int_{X_{1^3,3}} c_2 \wedge \sigma_1^2 = 198, \quad \kappa = 42,$$

$$h^{1,1} = 1, \quad h^{2,1} = 0, \quad h^{2,2} = 924, \quad h^{3,1} = 219, \quad \eta_{mn} = \text{diag}(42, 14c^2), \quad (\text{A.27})$$

$$\mathcal{D} = (\Theta - 1)\Theta^5 - 3z(3\Theta + 2)(3\Theta + 1)(2\Theta + 1)(13\Theta^2 + 13\Theta + 4)\Theta$$

$$- 27z^2(3\Theta + 5)(3\Theta + 4)^2(3\Theta + 2)^2(3\Theta + 1). \quad (\text{A.28})$$

$X_{1^2,2^2} \subset G(2,6)$:

$$\begin{aligned} \chi &= 888, \quad \int_{X_{1^2,2^2}} c_3 \wedge \sigma_1 = -344, \quad \int_{X_{1^2,2^2}} c_2 \wedge \sigma_1^2 = 208, \quad \kappa = 56, \\ h^{1,1} &= 1, \quad h^{2,1} = 0, \quad h^{2,2} = 604, \quad h^{3,1} = 139, \quad \eta_{mn} = \text{diag}(56, 42c^2), \end{aligned} \quad (\text{A.29})$$

$$\begin{aligned} \mathcal{D} &= (\Theta - 1)\Theta^5 - 4z(2\Theta + 1)^3(13\Theta^2 + 13\Theta + 4)\Theta \\ &\quad - 48z^2(3\Theta + 4)(3\Theta + 2)(2\Theta + 3)^2(2\Theta + 1)^2. \end{aligned} \quad (\text{A.30})$$

$X_{1^5,2} \subset G(2,7)$:

$$\begin{aligned} \chi &= 846, \quad \int_{X_{1^5,2}} c_3 \wedge \sigma_1 = -364, \quad \int_{X_{1^5,2}} c_2 \wedge \sigma_1^2 = 252, \quad \kappa = 84, \\ h^{1,1} &= 1, \quad h^{2,1} = 0, \quad h^{2,2} = 576, \quad h^{3,1} = 132, \quad \eta_{mn} = \text{diag}(84, 6c^2), \end{aligned} \quad (\text{A.31})$$

$$\begin{aligned} \mathcal{D} &= 9(\Theta - 1)\Theta^5 - 6z(310\Theta^5 + 919\Theta^4 + 884\Theta^3 + 476\Theta^2 + 132\Theta + 15)\Theta \\ &\quad - 4z^2(21311\Theta^6 + 78951\Theta^5 + 154395\Theta^4 + 180544\Theta^3 + 121086\Theta^2 + 42546\Theta + 6048) \\ &\quad - 8z^3(2\Theta + 1)(57561\Theta^5 + 249372\Theta^4 + 412273\Theta^3 + 310581\Theta^2 + 104388\Theta + 11691) \\ &\quad - 16z^4(2\Theta + 3)(2\Theta + 1)(10501\Theta^4 + 20138\Theta^3 + 13096\Theta^2 + 2676\Theta - 154) \\ &\quad + 1184z^5(2\Theta + 5)(2\Theta + 3)(2\Theta + 1)(\Theta + 1)^3. \end{aligned} \quad (\text{A.32})$$

$X_{1^8} \subset G(2,8)$:

$$\begin{aligned} \chi &= 636, \quad \int_{X_{1^8}} c_3 \wedge \sigma_1 = -336, \quad \int_{X_{1^8}} c_2 \wedge \sigma_1^2 = 300, \quad \kappa = 132, \\ h^{1,1} &= 1, \quad h^{2,1} = 0, \quad h^{2,2} = 436, \quad h^{3,1} = 97. \end{aligned} \quad (\text{A.33})$$

$X_{1^4,2} \subset G(3,6)$:

$$\begin{aligned} \chi &= 828, \quad \int_{X_{1^4,2}} c_3 \wedge \sigma_1 = -360, \quad \int_{X_{1^4,2}} c_2 \wedge \sigma_1^2 = 252, \quad \kappa = 84, \\ h^{1,1} &= 1, \quad h^{2,1} = 0, \quad h^{2,2} = 564, \quad h^{3,1} = 129, \end{aligned} \quad (\text{A.34})$$

$$\begin{aligned} \mathcal{D} &= \Theta^5 - 2z(2\Theta + 1)(65\Theta^4 + 130\Theta^3 + 105\Theta^2 + 40\Theta + 6) \\ &\quad + 16z^2(4\Theta + 5)(4\Theta + 3)(2\Theta + 3)(2\Theta + 1)(\Theta + 1). \end{aligned} \quad (\text{A.35})$$

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